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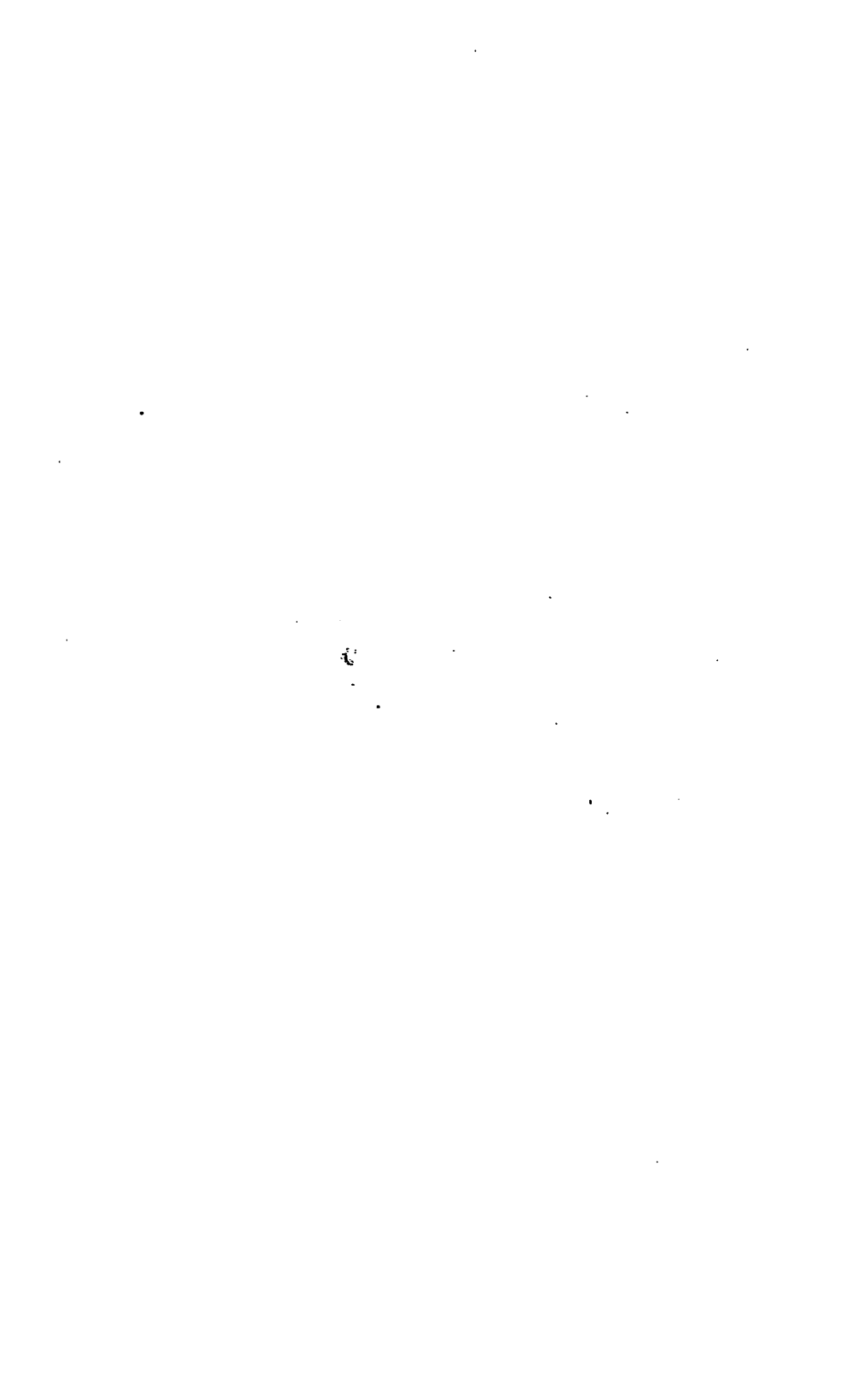
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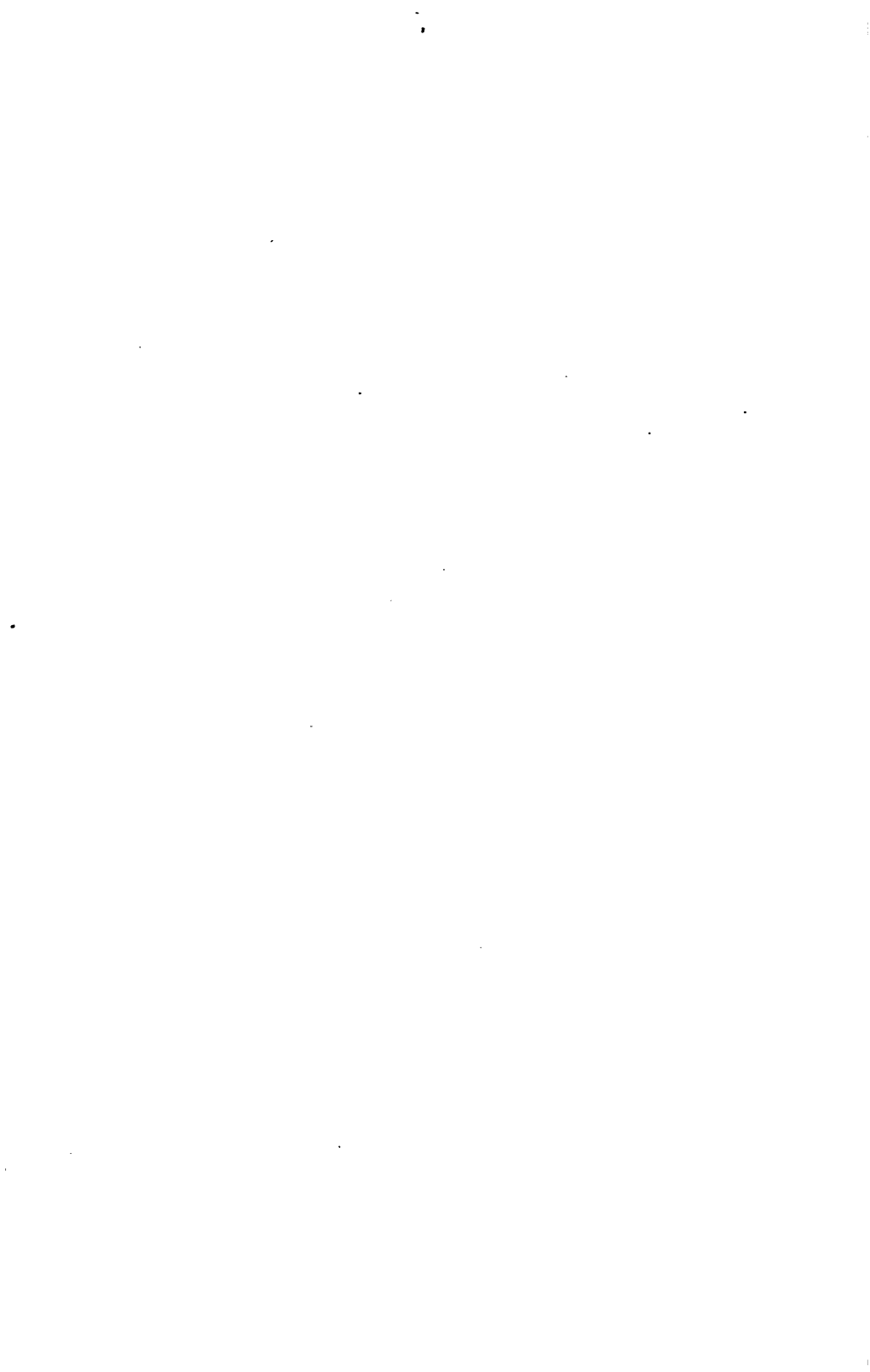
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A SERIES OF FIGURES
ILLUSTRATIVE OF
GEOMETRICAL OPTICS

REDUCED FROM STEEL ENGRAVINGS EXECUTED BY F. ENGEL
UNDER THE DIRECTION OF PROFESSOR K. SCHELLBACH OF BERLIN;

TOGETHER WITH AN EXPLANATION

FORMING

A TREATISE

TRANSLATED FROM THE GERMAN OF PROFESSOR SCHELLBACH.

The whole Edited,

WITH NOTES AND AN APPENDIX,

BY

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PREFACE.

THE Plates, which are the principal feature of this work, are lithographed copies of reduced drawings from a beautiful series of steel engravings which appeared last year in Berlin.

The description of the Plates is translated by the Editor's friend and pupil Mr. Tomkins of St. Catharine's Hall, from the German of Professor Schellbach. The notes and Appendix were added by the Editor.

Every Plate has been in a greater or less degree altered, chiefly for the purpose of making them less intricate and so better fitted for purposes of explanation, and more intelligible to a beginner. With this view some of the representations of eyes in different positions have been left out, as will be seen by referring to the notes. It appeared to the Editor that they might be dispensed with, because no new principle would be exemplified by retaining them. In the second and fifth Plates, and in figure 2 of the fourteenth, one-half of the original figures have been omitted; because the caustics are symmetrical with regard to the axis of each figure, and both sides are formed in precisely the same way. The modified figures shew the actual formation of the caustic on one side of the axis; and on the other side they indicate the way in which two rays intersect upon the caustic. In figures 1, 3, and 5 of the first Plate, two out of the three pencils proceeding from *A*, *B*, and *C*, are represented only in outline, because no

new principle is illustrated by giving them in full like the pencil from *A*. As a general rule one-half of the rays have been omitted from the figures which represent the modifications of pencils of light. The representations are thus relieved of much of their apparent complexity, whilst the graphic effect is undiminished.

The Appendix contains matter either important or likely to be generally interesting. At some future time its dimensions may be enlarged. At present the Editor does not think it necessary to add more; and he does not believe that there is room for a new treatise upon Optics.

It is hoped that the work will prove useful to English students. The original drawings, though unquestionably superior to these, were so expensive as to be beyond the reach of many students. The explanation also was in German: and this circumstance still more narrowly limited the number of persons who could derive benefit from the original work. A correspondence has been entered into with the Proprietors of that work, in order that they may be indemnified from any injury which the publication of this translation might entail; and it is believed that this object has been accomplished.

S. Catharine's Hall, June 3, 1851.

ERRATA.

Page 10, line 11, for *n* read *a*.

" 11, " 9, omit the word *about*.

GEOMETRICAL OPTICS.

EXPLANATION OF THE PLATES.

THE object of these drawings is to facilitate the difficult study of Optics, and especially to give a clear idea of the working of Optical instruments.

That the common treatises, and even careful calculations, do not effect this object in the same degree as constructions like these, we have the testimony of men eminent in physical science, even if a bare inspection of these drawings did not at once confirm this statement.

For the explanation of them we suppose only an acquaintance with the well-known laws of reflexion and refraction.

PLATE I.

Figure 1.

THE line LM represents the intersection of a reflecting plane with the plane of the paper at right angles. About this mirror is a perfectly uniform medium, within which, as is well known, rays of light are propagated in straight lines.

From the point A of this medium proceeds a pencil of rays, as AL , AG , AM , which are reflected from the mirror at the angle at which they are incident.

Thus, for instance, if the angle $ALG = OLN$, the ray AL , after reflexion, takes the course LO . The perpendicular ray AG is the only one which returns back upon itself, the rest after reflexion take a different course.

But if we suppose any one of these rays as LO , produced backwards, in the direction LA' , it is manifest, from the similarity of the triangles ALG and $A'LG$, that all the reflected rays so produced pass through a point A' , and thus return into the medium in the same way as if they proceeded from A' , a point at the same distance behind the mirror as the luminous point is before it. If we now conceive the entire figure to revolve about the line AA' , perpendicular to LM , then LM passes over the surface of the mirror, and the lines AL , AM describe a cone whose vertex is A , while $A'L$, $A'M$ describe a cone whose vertex is A' .

If then the luminous point A fills the cone ALM with rays of light, all these rays will be reflected as if they proceeded from the vertex A' .

Suppose now an observer at E to look upon the mirror, his eye receives from it a small cone of rays whose vertex is A' , and whose base rests on the transparent cornea of the eye: here the rays undergo refraction, pass through the fluids of the eye, and after various other refractions converge upon the retina to the vertex of a cone, where they have sufficient power to stimulate

the optic nerve so as to give rise to the perception of sight. Thus the eye is affected just as if the rays had proceeded from the point A' , and therefore an observer at E not only sees the point A in the direction EHA' , but believes he sees it in the point A' itself.

The point A' may thus be called the *image* of A .

In speaking of a luminous point, it need scarcely be said that we do not mean a *mathematical* point; such a point would be no more visible than a mathematical line. By a luminous point we understand a minute portion of a luminous surface in no direction perceptibly extended. And so by a luminous line we understand such a surface perceptibly extended only in one direction.

In figure 1 two other points B, C are taken from which pencils of light proceed. The images of these points are observed in B', C' ;* these images A', B', C' are all seen in the same position by any eye before the mirror. But this simplest case happens only when the mirror is plane.

Figure 2.

In this figure A, B, C is supposed to be a luminous object whose two extremities and middle point have the same position relative to the mirror as the points A, B, C in figure 1; the rays AH from A , BI from B , and CK from C , come to the eye at E . Or rather the eye at E receives from the points A, B, C pencils of rays of which HE, IE, KE are the axes, and whose vertices would lie in A', B', C' , if produced beyond the mirror.

But from all points of the arrow ABC such cones of rays reach the eye, and therefore it sees an image of this arrow in A', B', C' . In like manner, eyes at D and F manifestly see the object ABC in exactly the same form, in the position $A'B'C'$, and at the same distance behind the mirror as the arrow is before it.

Figure 3.

In this figure we suppose the space below the line DE to be occupied by water, the space above by air.

* In the original plate all the rays from B' and C' are also drawn. The greater part are here omitted to avoid confusion.

From the luminous point A proceeds a pencil of rays DAE , which is refracted on emerging into the air.

Thus let AK be a ray of light, and AH a perpendicular to the surface DE , and suppose a normal drawn at K . Then KAH is the angle which the ray makes with the normal, i.e. it is the angle of *incidence*. If now the refracted ray KL be produced backwards to cut the perpendicular AH in O , then the angle KOH represents the angle of *refraction*.

The refractive index for water is nearly $\frac{4}{3}$; so that we have $\sin KAH : \sin KOH = 3 : 4$; or, since in the triangle KAO these sines are as $KO : KA$, we have $KO : KA = 3 : 4$. If then KA be divided into four equal parts, and with centre A and radius equal three of these parts a circle be described, it will cut the perpendicular in O . In order therefore to find the direction of the ray AK after refraction, we have only to produce the line OK to L .

In this way all the rays which emerge from the water may be accurately determined.*

If all the rays proceeding from A , which fall between the legs of the angle DAE , could have been constructed, the emerging refracted rays produced backwards into the water, would have formed by their mutual intersection a continuous curve which is even here to be perceived in the figure $DMA'E$, and appears more fully developed in figure 4.

This curve to which the refracted rays are tangents, is one of the simplest kinds of *caustics*, which perform so important a part in Optics.

From the above construction of the refracted rays it appears also that the cusp A' of this caustic line lies $\frac{1}{4}$ th part of the distance AH of the luminous point from the surface of the water higher than this point; for a ray incident close to the foot H of the perpendicular AH , will manifestly be so refracted as to intersect in A' if $HA' = \frac{3}{4}HA$. It is also manifest that the caustic line only extends to the points D and E , if $DH = EH = \frac{3}{4}DA$, for then the ray which emerges at D or

* The author of the original work here remarks that "every single ray in these figures has been constructed with such care that the angular error is never greater than a few minutes."

E carried backwards intersects in H , and thus remains constantly in contact with the surface.

We know that every ray which reaches the separating surface of two media is there separated into two parts, of which the one enters by refraction into the new medium, the other is reflected into the old one.* All rays from A falling upon the line DHE undergo this separation, though in the figure the refracted rays only are given. But if the rays are incident on such points as lie beyond D and E , they no longer make their way out of the water, but undergo only one deflexion. As, for instance, the ray RA will be reflected towards RS , and an eye situated in the water at S would perceive a very bright reflected image of the point A , in the direction SR . The reason why this image would be so bright and distinct is that the entire ray is reflected, and no portion of it is lost by refraction, as in the case of those rays which are reflected from the part DE of the surface. On this account rays from A , incident beyond D and E , are said to undergo *total reflexion*.

If we now suppose an eye situated at L , this eye in addition to the ray KL receives a large portion of those in the neighbourhood of it. These rays do not indeed form a cone about KL as an axis; they will however enter the eye in a body as if their origin lay very near the point M , in which the ray LKM touches the caustic line; they must therefore produce an effect on the eye like that which would be produced by a cone of light with vertex M . So that the eye sees the point A , not only in direction LKM , but as if it were in the point M itself, or at least in a position differing exceedingly little from that point.†

Hence it is easy to understand why an observer at I will see the point A at another position A' , though in the same direction IHA . As also why the bottom of a vessel filled with

* Besides the separation of the ray into two parts, which follow the common laws of reflexion and refraction, there is a third part which is dispersed at the surface. This *scattered* light, as it is called, follows no law, but proceeds from every point of the surface as it would do from a luminous point. This light renders the surface itself visible to the observer.

† The pencil in fact is a small oblique pencil, which proceeds from two focal lines and a circle of least confusion. The point M is the *primary focus*.

water must appear to be raised about a fourth part of its distance from the surface.

If we suppose the entire figure to revolve about the normal IA as an axis, the line DE describes the surface of the mirror, the angle DAE a cone which we are to suppose occupied by rays proceeding from A as a vertex, and the caustic line $DMA'E$ will then generate a cone-shaped surface of revolution which may be called the caustic surface; because from it all the rays which emerge from the water will appear to proceed.

To any eye which is above the water, the luminous point A can appear only in some one point of the caustic surface. It is known to mathematicians that the caustic lines which are formed by refraction at a plane, are the *evolutes* of conic sections. That which is here represented is the evolute of an ellipse.

The engraving shews two other luminous points B, C with their caustics, which will be made use of for the following figure.*

If we suppose below the surface DE a medium different from water, as glass, the drawing will still be essentially the same, the extremities D and E of the caustic lines in that case only lying nearer the point H , and the vertex A' will be elevated by about $\frac{1}{3}$ of the depth HA , as the index for glass is about $\frac{3}{2}$.

Figure 4.

This figure will now be easily understood, as in it the same things are indicated by the same letters as in figure 3. The arrow ABC is in the water; every one of its points sends a pencil of light to the surface DE , the rays of which are there refracted in the same way as if they came from the caustic surface corresponding to the luminous point. Thus the eye at L , receiving from the point A the ray AKL , sees this point on the caustic line in M , and so from the point B the ray BPL reaches the eye and *transfers* the image of B towards N where LPN touches the corresponding caustic. The point C can be seen by the eye only in the point O , where the ray CQ refracted towards QL touches the caustic of the point C . If we now suppose caustics

* In the original figure *all* the rays from B and C are also drawn.

constructed from every point of the arrow ABC , this rectilinear object will appear to the eye as a curved line MNO .

The figure represents three other eyes,* and determines in a manner easily understood the positions in which an image of the object ABC appears to them.

Figure 5.

If figures 3 and 4 have been comprehended, the following brief account of the next figure will be sufficient. Below the line IG is water, above it air. From the point A a pencil of light proceeds which meets the watery mirror IG , and after refraction enters the fluid. If the refracted rays are produced backwards into the air, they form a caustic line by the continued intersection of immediately successive rays. Thus if we suppose AD , AD' two such rays, which after refraction take the course DE , DE' , these meet in a point d on the caustic line, and an eye in the neighbourhood of EE' would receive a pencil of rays whose vertex would appear to lie in this point. It must thus see the point A in d .

Suppose now all the rays constructed which fall from A on the indefinitely extended surface IG , the refracted rays produced will form the complete caustic of the point A , whose two branches ae , af then extend to infinity. This caustic is known to be the evolute of the hyperbola. The figure shews two other pencils of light whose vertices are in B and C , and their caustics in b and c .†

Figure 6.

The luminous object is here again the arrow ABC ; the pencils from these three points with their corresponding caustics are delineated in the preceding figure. In this the caustics only are given, and of the rays only those which fall upon the eyes G and G' .

Thus the ray AD from the point A , after being refracted into the direction DG , falls upon the eye, and being produced,

* In the original figure two more eyes are drawn, which it has not been thought necessary to retain.

† The rays in these pencils are omitted for the sake of simplicity.

touches the caustic of the point A in d . But AD may be considered as the axis of a small pencil of rays which reaches the eye, and which after refraction, though no longer in fact a cone, falls upon the eye in a cone-like form. The rays of the pencil produced backwards will not converge to a point, but will be crowded together almost to a point in d , and the effect will be the same upon the eye as if the rays proceeded from a luminous point in d , i.e. the eye will see the point A in d .

It is also now evident in what way to the eye G , the point B must appear in the point e of the caustic b , and the point C in point f of the caustic c .

If the caustics of all points of the object ABC were constructed, a very accurate image of the arrow might be traced as it appears to the eye G .

By the help of several of these caustics the image def has been carefully drawn. The eye G' manifestly sees the arrow in position $ae'f'$; i.e. because the ray AD reaches the eye G' without refraction, it sees the point A , only at a greater distance, in the vertex a of its caustic.

From what has been said, it is now sufficiently evident where and in what form an object must appear, situated in a medium whose refraction is less than that of the medium from which the object is observed.

PLATE II.

The luminous point C throws rays upon the circle whose centre is M and diameter AB ; the circumference is supposed to be a reflecting surface. A circle, as every other curve, may be regarded as a polygon with an infinite number of sides, or as the limit to which such a polygon approaches when the number of its sides is indefinitely increased; one of these indefinitely small lines produced is then a tangent.

If now a ray Cl^* from C falls upon the circle, we may

* In the original figure all the rays from C are drawn. In this figure those rays only are drawn which fall on the lower semicircle; and the caustics only which are formed by the rays reflected from the other semicircle.

assume that it meets the bisection of one of these indefinitely small lines, to which the radius MI is perpendicular.

If therefore lm be the reflected ray, the angle $ClM = Mlm$. According to this law all the rays of the figure are accurately constructed. If we could draw the ray immediately succeeding Cl , which may be supposed to be Cl' , these two rays would intersect in a point n ; and if all the intersecting points of immediately successive rays were constructed, we should obtain all the points of the caustic idc, d_1i_1 .

A caustic formed by reflexion is called a *catacaustic*, to distinguish it from one formed by refraction, like the two already considered, which is called a *diacaustic*.

The portion of the rays coming from C which fall on the arc fAf_1 , strictly come to mutual intersection on the curve idc, d_1i_1 ; so that, if for example a spring of polished steel be bent into a circular arc fAf_1 , and a very small luminous body be applied at C , a bright line appears on the paper which has very nearly the form of the catacaustic. But in general the whole space between this and the spring will be as much more illuminated than the rest of the paper, as this part in our figure appears more dark with intersecting lines than the rest.

The rays which coming from C fall upon the arc fBf_1 , will be reflected in the same way; but these reflected rays are indicated only by short lines, as they never again mutually intersect, and would, if carried out, have interfered with the part of the caustic to the left of C which really comes into view. On the other hand, these reflected rays have been carried backwards beyond fBf_1 , and have there, by mutual intersection, formed another caustic $hc'h$.

Cr and Cr_1 are intended to represent two immediately successive rays such as fall from C on the arc fBf_1 . After reflexion they become more and more divergent, and extend towards p and p_1 . But if they are produced to the right they will intersect in s , and then proceed towards t and t' .

The legs $c'h$ and $c'h'$ of this *subjective*, or as it is called virtual part of the caustic extend to infinity, as do also the branches id, i_1d_1 , of the *objective* part. Both parts together constitute the complete caustic of the point C . The straight lines kfg

and k, f, g' are to this curve what mathematicians call *asymptotes*, that is, straight lines to which id and i_1d_1 as well as hc' and $h'c'$ continually approach without ever meeting them, so that the distance between them and the curve vanishes only at infinity.

Let the diameter AB of the circle be the axis of abscissas, the centre M the origin of coordinates, and the positive abscissas (x) reckoned from M towards B , the perpendiculars upon it positive ordinates (y). Put r = radius, $CM = a$ the distance of the luminous point from the centre, and we shall find for the caustic the following equation

$$\{(4a^2 - r^2)(x^2 + y^2) - 2ar^2x - a^2r^2\}^2 = 27n^2r^4y^2(x^2 + y^2 - a^2)^2.$$

If in this equation $y = 0$, we have the position of the two points c' and c_1 of the caustic given by the expressions

$$Mc' = \frac{ar}{2a - r}, \quad \text{and} \quad Mc_1 = -\frac{ar}{2a + r};$$

or if we put $b = Mc'$ and $c = \frac{1}{2}r$,

we have $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, and the expression for Mc_1 is obtained from that for Mc' by putting $-r$ for r , by which we have

$$\frac{1}{a} + \frac{1}{b} = -\frac{1}{c}.$$

The two cusps c' and c_1 are formed from the two opposite sides B and A of the reflecting circle on which account c has opposite signs in the two expressions.

If we are only inquiring into the effect of one side of the circle on the rays of the luminous point C , the first of these equations will obviously be sufficient; and this equation remains unaltered if a and b no longer indicate the distances of C and c' from the centre, but the distances of these points from B .

If now we put the factor $x^2 + y^2 - a^2 = 0$, we obtain the coordinates of the points d and d_1 ; those for d are

$$x_1 = \frac{a}{r^2}(2a^2 - r^2) \quad \text{and} \quad y_1 = \frac{2a^2}{r^2}\sqrt{(r^2 - a^2)}.$$

The cusp d_1 has the same abscissa and an equal but opposite ordinate. These cusps are thus at the same distance from the centre M as is the luminous point C .

If we determine the tangent at the cusp d , it is obvious that it is formed by the reflexion of the ray Ce , which is perpendicular to MC . This cusp therefore is at the same distance from the point e as the point C .

If we put $m =$ the angle fQB which the asymptotes kg, k_1g' make with the axis Mq , we have

$$\sin m = \frac{1}{ar^2} \left(\frac{4a^2 - r^2}{3} \right)^{\frac{1}{2}},$$

and the distance of the point q , in which the asymptotes meet the axis, from the centre, or Mq , is about $\frac{3ar^2}{4a^2 - r^2}$.

The expression for $\sin m$, when $a < \frac{1}{2}r$, will be imaginary. Therefore the curve has asymptotes only when the luminous point is further from the centre than half the radius.

It may now be clearly seen how the arc ff_1 affords that part of the caustic which falls to the right of B , and which extends indefinitely from c' towards h and h' . The arcs fe, f_1e_1 give rise to the branches di, d_1i_1 of the caustic, which are also indefinitely extended, and the arc eAe_1 forms the finite part dc, d_1 of this curve.

In the figure $MC = \frac{3}{4}MB$. If therefore $r = 1, a = \frac{3}{4}$, and we have $Mc' = \frac{3}{2}, Mc_1 = -\frac{3}{10}, x_1 = \frac{3}{32}, y_1 = \frac{9\sqrt{7}}{32} = 0.744,$
 $Mq = \frac{9}{5}, \sin m = \frac{5\sqrt{15}}{54} = \sin 21^\circ 1'.$

As the equation of this catacaustic is of the sixth degree which reduces to the third for the ordinate y only, the form of the curve could not be obtained from its equation without considerable difficulty, though the distinctive points of it are found pretty easily.

A perfectly clear notion of the effect of a reflecting sphere on rays which fall from a luminous point, may be obtained by supposing our figure to revolve about the line MC as an axis.

The reflecting circle then describes a sphere illuminated from the point C , and the caustic curve forms by this revolution the caustic surface in a manner very intelligible. The cusps c , and c' of the catacaustic curve then form vertices of the catacaustic surface, but the points d and d_1 give rise only to an edge of this surface.

PLATE III.

In this plate the foregoing constructions have been applied to explain the effect of a *concave* spherical mirror. The arc *DBE* is supposed to be reflective, and every point of the luminous object *aCb* gives rise to a caustic. Three of these only have been given, viz. those which belong to the points *a*, *C*, and *b*, and these lines are all exactly similar, as the arrow *aCb* is only an arc concentric with the mirror, and thus all its points are at the same distance from the centre of the reflecting circle, on which alone the form of the caustic depends.

The caustic *C'c*, *C''* belongs to *C*, the middle point of the arrow, the line *a'a''* belongs to the extremity *a*, and the line *b'b''* to the other extremity *b*.

It will at once be seen, by carefully considering Plate II, that an eye *A*, which looks from the centre of the mirror towards *B*, receives pencils of rays whose vertices lie in *a*, *b*, *c*, and will therefore see in these points an inverted image entirely similar to the object. As the visual angle $a_1 A b_1 = \angle a A b$, an eye at *A*, will see the object and image of the same size, though in the figure the latter is represented smaller. An eye *A''* at a greater distance would see the image still in nearly the same place, but under a smaller angle and therefore of diminished form. An eye at *A*, would receive from the object the rays *ad*, *Ce*, *bf*, which all arrive at *A*, after reflexion. These reflected rays touch the caustics in *a'*, *c'*, *b'*, and thus, for reasons repeatedly stated, the image of the object *aCb* appears in the form and position of the figure.

The figure represents two other eyes *A'''* and *A''''*,* with the images of the object as seen by them. It is obvious that it is only from a concentric arc as *aCb*, that an image similar to the object can be formed in the cusps of its caustics, and that the image of an object of any other form must appear more or less distorted, as the distance of the cusp of the caustic from the centre of the reflecting circle is not proportional to that of the luminous point from the centre; for if the latter is *a*, the former is $\frac{ar}{2a + r}$.

* The eye *A'''* has been omitted.

A spherical concave mirror is a segment of a sphere which reflects from its concave surface. The centre A , of the sphere is called the centre of curvature of the mirror, and the centre B of the spherical segment is called the centre of the face of the mirror. If a plane be made to pass through the eye of an observer at A , the centre of curvature A , and a luminous point C of the object, so as to cut the object in the line aCb , then $a_c b_c$ is obviously the image of this section as seen by the eye. If the plane through the eye and centre of curvature pass through other points of the object, the images corresponding to the sections of the object are those given in the figure.

Scarcely one, however, of the images will be like the corresponding section, therefore in a concave mirror the whole object will in general appear only in a distorted form; and if this distortion is not always apparent, the only reason is that the observer seldom forms an accurate judgment of the relative distances of the different points of an image. The images which the figure represents have to the eye the appearance of hanging in the air before the mirror, and one of them, $a_1 b_1 c_1$, is further distinguished by the circumstance that it can be made to appear by being received on a colourless opaque screen. If a white sheet of paper, for instance, were to be applied at the points a_1 , b_1 , c_1 of the caustics, every point of the object would illuminate a point of the paper with its own colour, and thus the entire object would be depicted on the paper. A clear idea of the form of the caustic surfaces corresponding to the caustic curves will make it evident that such a picture can be formed only in the cusps of the caustics given in the figure; for a sheet of paper a little further to the left of these cusps would manifestly intersect the caustics in circular sections, so that to every point of the object would correspond on the paper a circle instead of a point; and these circles would cross and interfere with each other so as to produce only a confused image of the object.

The expression $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, where c = half the radius, a = the distance of the luminous point from the mirror, and which gives the distance of the image of that point, is not altered if a and b be put for each other; or, which is the same thing, a

luminous point at distance a projects its image at distance b , and conversely a luminous point at distance b has an image at distance a . Hence it is obvious in what way a small object $a_1b_1c_1$, by means of a concave mirror, presents a magnified image aCb , which may be received on a white screen.

PLATE IV.

The preceding Plates represented the *objective* images of the concave mirror, and the special object of this is to explain the *subjective*, which appear behind the mirror in the same way as in the common plane reflector.

The object is here again the arrow aCb , and the figure shews the effect of the three principal points on an eye looking towards the reflecting arc DBE . The pencil which comes from a and meets the circumference in g , is reflected as if it came from a' behind the mirror. Therefore an eye in the centre A' of the circle, sees an image of the point a behind the mirror at a' . This point a' is the cusp of the caustic which belongs to a , and which is represented in the figure by a dotted line. In like manner the same eye will see the points C and b in c' and b' , so that from the whole object in general, an upright and magnified image will be seen behind the mirror in the position $a'c'b'$. It can only however seem to be magnified in case the eye can judge of its distance; for instance, the visual angle $a'A'b'$ has continued $= aA'b$, so that the object and its image appear to be of the same size. It will be understood from what has been said in explanation of preceding figures, in what way an eye at A'' sees an image of aCb , viz. $a''c''b''$ behind the mirror, but no longer exactly similar to it. If the reflecting circle is of sufficient size, an eye sees from the point A''' , three images of the object; viz. a *subjective* one $a'''c'''b'''$ behind the mirror, and two *objective* ones $a_{,,,}b_{,,,}$, a_2b_2 , suspended before the mirror. The subjective image is formed by such pencils as ad' , Ce' , bf'' ; and the pencils ad'' , Ce'' , bf''' , and ad''' , Ce''' , bf'''' after reflexion produce the two objective images.

The figure also gives other images of the object as seen by the eye A'' .* From what has been said it will readily be seen how these constructions illustrate the effect of a concave mirror. As the concave mirrors generally used consist of a small portion only of the spherical surface, the two objective images before the mirror which arise from the reflexion of the rays coming from the parts nearest the rim of the mirror, are seldom seen.

PLATE V.

Figure 1 of this plate represents the caustic of the circle when the luminous point C is only $\frac{1}{4}$ of the radius from the centre M . The curve which before extended on both sides to infinity, is here compressed into the narrow space dcd_1c_1 . Since here $MC = \frac{1}{4}$, if we put radius = 1, we have, according to the formula given in the explanation of Plate II, $Mc = -\frac{1}{6}$, $Mc_1 = -\frac{1}{2}$; the abscissa of the cusps d and d_1 is $-\frac{7}{32}$, and the two ordinates are $\pm \frac{\sqrt{15}}{32}$ or ± 0.121 . The asymptotes have vanished, as the distance MC is now less than half the radius. The figure will now explain itself: it will be only necessary to follow the individual rays which come from C , in order to perceive how after reflexion they form the caustic.

Figure 3 represents the caustic which parallel rays, such as kl , kl' , form by reflexion from the semicircle DAE . This caustic is called the *cardioid*.† It is that curve which a point in the circumference of a circle describes, which rolls on a circle of twice its own radius and is thus an epicycloid; the fixed circle on

* This eye is omitted.

† This curve is not usually called the *cardioid*; that name being confined to the epicycloid which is generated by the rolling of a circle upon a circle of equal radius. The caustic assumes the form of the *cardioid* in this restricted sense when the luminous point is at the extremity of the diameter of the circle.—(See Lloyd's *Optics*, pp. 69, 71. London, 1831.)

which the other rolls is here described with centre M and half the radius of the circle DAE ; the equation of this curve is obtained from the general expression in page 10, by putting $a = \infty$. It then takes the form

$$\{4(x^2 + y^2) - r^2\}^3 = 27r^4y^2.$$

The figure shews how strongly the rays of the caustic are concentrated in the neighbourhood of the point c_1 , and how, by means of a spherical concave mirror, a considerable degree of heat may be developed in the focus c_1 , so that such a mirror is rightly called a *burning mirror*. As it was this property which first and chiefly attracted the attention of philosophers, the cusps of caustics have been called *foci*, and the name *caustics* has arisen.

In figure 2 the parallel rays have been made to reflect immediately from the semicircle on which they first fall. The rays so reflected become, to the right, more and more divergent, and would form a caustic only in the circle by being produced to mutual intersection. And in this way the caustic in the figure is formed. It agrees entirely with that formed in figure 3, or rather together with that completes the catacaustic produced by parallel rays reflected from a circle.

With regard to figure 4, it needs only to be said that it represents the caustics which rays from the luminous point C , after reflexion from the further side AB of the mirror, would form within it.

PLATE VI.

This plate represents the distorted objective images which the concave mirror DBE would shew of the object acb , to observers in A_1, A_2, A_3, A_4 . To perceive how these images are formed, it is only necessary to take Plate V, and trace the way in which each of the three caustics there are formed, which belong to the points a, b, c of the object.

Many more caustics than the figure shews, have been employed to delineate the form of the images, which have thus been very accurately drawn. To an observer, indeed, these images do not appear so out of shape as the figure represents them, because the eye is unable to judge accurately whether a visible point is nearer or more distant.*

PLATE VII.

Figure 1.

In figure 1 of this plate the object is again a circular arc, described from the centre of curvature of the mirror, but it is placed at the distance of $\frac{1}{2}$ of the radius *beyond* this centre, instead of being, as in Plate VI, so much nearer the mirror. The objective images which the eye now sees in the air, appear also inverted, but diminished and less distorted than those in that plate which represents them magnified.

Figure 2.

Figure 2 shews the images of an object *aCb*, much diminished and in an upright position, which a *convex* mirror would form. These images appear to the observer behind the reflecting surface like those of a plane mirror, except that they are always smaller than the object, and their position changes with that of the observer.

Figure 3.

The object in figure 2 was an arc described from the centre of the reflecting surface, and all the caustics arising from the several points of this arc were similar and equal. But in figure 3 the object is rectilinear, and thus the caustics which belong to its different points are no longer similar and equal, since these points are at different distances from the centre of the mirror. Accordingly the images in the figure are perceptibly different.

* In the original plate two more eyes are represented.

PLATE VIII.

Figure 1.

The triangle DEF represents the intersection of a triangular right *prism* with the plane of the paper, and its faces are perpendicular to this plane.

From the luminous point c falls a pencil of rays cDv , on the side DE of this section of the prism, and becomes refracted on entering the *glass* medium. The index for glass has been taken 1.5. The refracted rays, if carried backwards into their former medium, form there a caustic which will be understood from the remarks on Plate I. At the side DF of the prism these rays are again refracted; but those only which fall between D and l are able to emerge from the glass; the rest, which strike the prism between l and F , are totally reflected, as appears from the explanation of Plate I. The rays which come out from the line Dl , are after refraction produced backwards so as to form the caustic Cil . If the course of two rays next each other, as $cefg$, $ce'f'g'$, be traced after the second refraction, which takes place at ff' , their mutual intersection is on the caustic in i . And thus an eye which receives the pencil $fgf'g'$, at $abgg'$, would see at i the image of the luminous point c . The line $rcsu$ shews the direction of the ray perpendicular on DE , and which therefore passes through DE unrefracted, and is first refracted at s . The ray ckl divides the entire pencil into two parts, of which only that between D and l comes into the air, while the other, which falls between l and F , is thrown back into the prism by total reflexion, and can emerge only at the side FE between F and m . Two of these rays, supposed consecutive, as mn and $m'n'$, carried backwards intersect upon a caustic represented by pC' . The drawing shews also the caustics A and B in outline, which belong to the pencils aDx and bDy , proceeding from the points a and b . In like manner A' and B' indicate the caustics which correspond to the totally reflected parts of these pencils, emerging from the side EF of the prism. To the point c belongs in fact not a caustic line, but a very complicated caustic surface, of which the figure shews only a section made by a plane through the luminous point perpendicular to the faces of the prism.

Figure 2.

This construction is here made use of to explain the images which a prism DEF shews of the object acb . The figure gives only those caustics which belong to the extremities and middle of the object.

Now it is known that every ray, when it undergoes refraction, is separated into a multitude of rays, of which the red rays are the least refrangible and the violet the most. The pencil which falls on the prism from c is thus, after refraction, separated into a multitude of pencils of different coloured rays, and to every one of these pencils belongs a particular caustic.

The figure represents by the lines A, C, B , the two outermost of these caustics; by the dark lines, those which correspond to the extreme violet rays; by the fainter lines, which run to the right near the darker, those which belong to the red rays. The index for the red rays has been taken 1.50, and for the violet 1.53.

The six lines drawn through the vertex D of the prism correspond to the extreme rays from the points a, b , and c . The ray cd , when it enters the prism, is separated into rays of different colours, of which only the violet di is given. This violet ray comes again into the air at i unchanged,* and so reaches the eye at A_1 . But of the ray cf in the figure, only the red part fk is retained, which after its second refraction is received also unchanged by the eye A_1 . The lines iA_1 and kA_1 may be regarded as the axes of small pencils, whose vertices lie in the points c' and c_1 . In this position, then, two coloured images of the point c appear to the eye A_1 ; viz. a violet one in c' and a red one in c_1 . But these are only the two extreme images of the point c : between these a multitude of others would appear in accordance with the known gradations of coloured images.

In a similar way the images of a and b are shewn in $a'a_1, b'b_1$. It is now easily perceived how to the eye A_1 , a violet image of acb is seen in $a'c'b'$, and a red image in $a_1c_1b_1$. Between these lies a multitude of other coloured images arising from the separation of the other rays into colours.

* Without undergoing any further separation into partial colours.

In the middle all these coloured images are hidden*, and in accordance with known laws produce a colourless image, which however at a' and b_1 has a coloured border, and this from being violet at a' becomes white at a_1 , and passes from red to white between b_1 and b' .

Just such an image of the object, coloured at the edges, is seen by A_1 within the prism. The reason why an image seen through a prism is not so indistinct as would hence be supposed, since every point presents a multitude of images, is, that all these images nearly coincide, and because it is not a single ray but a pencil which generates an image.

Of the ray ch after refraction only the violet part hm is retained in the figure. But this ray at m undergoes total reflexion, arrives thus at n , and thence, by a second refraction, at the eye A_2 . Of the ray cg , on the other hand, the figure gives only the red part gl , which is also totally reflected at l , and is then refracted at o towards the eye A_2 .

From what has been said it is now obvious, how on the caustic C' to the eye A_1 a violet and red image of the point c must appear in c''' and c_s , and how in general it must perceive in $a'''c'''b'''$ a violet, and in $a_sb_sc_s$ a red image of the object acb . Images of other colours lie between these two which merge into a single image with a coloured border.†

PLATE IX.

This plate exhibits the course of the rays through a *spherical lens*. It is of special importance and requires to be carefully studied.

By a spherical lens is meant a transparent substance, which is bounded on two sides by spherical surfaces, through which the rays pass.

* They are super-posed throughout the central portion of the image, and so appear to blend into an indistinct white image.

† This is not a literal translation.

Figure 1.

In this figure, M and M' are the centres of the spheres CBC' and $CB'C'$: between them the lentil-shaped piece of glass $CBC'B'$ is contained. If the arcs BC and $B'C$ be supposed to revolve about BB' as an axis, they will obviously pass over the whole surface of the body; which surface we suppose to be filled with the refracting substance.

The line MM' , which joins the two centres of the spherical surfaces, is called the axis of the lens; and the figure thus represents only the section of the lens by a plane which passes through the axis. The two surfaces of the lens are here formed by two spheres of equal radius; they might, however, have different radii.

In the figure both surfaces are curved outwardly; and such a lens is called *double-convex*. If both were curved inwardly, the lens would be *double-concave*. If one surface were plane, the lens would be either *plano-convex* or *plano-concave*.* Two other lenses are called *concavo-convex* or *convexo-concave*, according to the form of the surface first considered.

In figure 1 A is a point in the axis of the lens from which rays fall upon the surface CBC' . It has been already explained on Plate I, figure 3, that a ray incident on the plane surface of a transparent mass, is separated into two parts, of which the one enters the medium, and the other is reflected. The manner of this reflexion and refraction in the case of a plane surface is already known: but when the surface is curved, it will be most convenient to suppose it covered with a vast number of minute plane triangles, whose magnitude may be conceived to be evanescent; so that in the limit the surface will not in any assignable degree differ from one of continuous curvature. Suppose now that every ray which falls upon a curved surface lights upon one of these small plane triangles, then its direction after reflexion and refraction may be determined as on Plate I, figure 3; the principal object being to

* These lenses, when inverted, are sometimes called *convexo-plane* or *concavo-plane*. The two lenses last mentioned in the text are in fact the same lens considered in inverted positions; and this lens is called a *meniscus*.

determine the direction of the perpendicular on such a triangle, which is called a normal to the curved surface. In the sphere these normals are manifestly no other than the radii of the sphere.

If, for instance, it is required to determine the direction ab which the ray Aa takes after entering the lens CC' , we must draw the radius Ma , and determine the direction ab , so that $\sin baM = \frac{3}{4} \sin AaM$.

If the ray ab comes again into the air at b , its direction bc is then found by drawing the radius $M'b$, and determining the angle $M'bc$, so that $\sin M'bc = \frac{3}{4} \sin M'ba$. And in like manner all the rays of the figure are drawn. The entire pencil of rays CAC' in this way passes through the lens, and extends behind it, so as to occupy the space $CHH'C'$. The boundaries CH and $C'H'$ of this space again form what is called a *diacoustic*; for these lines are formed by the constant intersection of successive rays which fall on the arcs aC , $a'C'$ of the lens.

But the rays which fall on the part aa' of the lens, no longer mutually intersect after emerging from it, but diverge, and can only intersect by being carried backwards. The figure shews these productions. They form by their mutual intersection another part $IA'I'$ of the caustic. The branches CH , $C'H'$, $A'I$, and $A'I'$, extend to infinity, and continually approach the asymptotes dc and $d'c'$.

It is of some importance to determine the position of the asymptotes with regard to the lens, for they divide it into two parts, which, as we shall see, produce effects essentially different.*

If we now suppose the whole figure to revolve about the axis EF of the lens, the angle CAC' passes over the surface of the cone of rays coming from A , and the lines CH , $C'H'$, describe the conical space into which the refracted rays extend.

So also the curved lines $A'I$ and $A'I'$ form by this revolution the cone-shaped portion of the caustic surface whose cusp is at A' , and which belongs to the pencil CAC' ; and the asymptotes form a conical surface which intersects the caustic surface in a circle

* This is not a strictly literal rendering of the original.

whose diameter is GG' . Through this circle all the rays pass which lie within the asymptotic cone.

If now we put r and r' for the radii of the spheres M, M' , and $a = AB$, the distance of the luminous point from the surface turned towards it; also $a' = B'A'$ the distance of the cusp A' from the second spherical surface, $l = BB'$ the thickness of the lens, n = refracting index, which is here taken = 1.5, from the rarer to the denser medium; if moreover $m = n - 1$, then a' is found from the equation

$$\frac{1}{\frac{m}{r} - \frac{1}{a}} + \frac{1}{\frac{m}{r'} - \frac{1}{a'}} = \frac{l}{n}.$$

In the figure the half-thickness of the lens is taken as the unit, and $r = r' = 6$, $a = 3$, $l = 2$, $n = \frac{3}{2}$, and $m = \frac{1}{2}$; hence we have

$$\frac{1}{\frac{1}{12} - \frac{1}{3}} + \frac{1}{\frac{1}{12} - \frac{1}{a'}} = \frac{4}{3}, \quad \text{or } a' = -9\frac{3}{4}.$$

The negative sign of a' shews that the point A' is to be sought, not like the point A , in the direct production of the radius $M'B'$, but by going backwards from B' through M' .

The formula here given avails for all the six kinds of spherical lenses, by giving proper signs to the different magnitudes.

In many cases the values of r, r', a, a' , are so considerable with respect to the thickness l of the lens, that this thickness may be neglected. In that case the formula becomes

$$\frac{1}{a} + \frac{1}{a'} = \frac{m}{r} + \frac{m}{r'},$$

or putting $\frac{m}{r} + \frac{m}{r'} = \frac{1}{F}$, it is

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{F};$$

which formula is exactly similar to that for the convex mirror.

This equation enables us to find a' when a and F are

given. If the luminous point A be supposed at an infinite distance, then $\frac{1}{a} = 0$, and $F = a'$, F is the distance of the focus from the lens, *i.e.* of the cusp of that caustic which would be formed if the rays falling on the lens were parallel to its axis.

In the lens here represented $F = 6$; therefore this point, if parallel rays were incident on the surface CBC' , would lie to the right of B' at a distance equal to $M'B'$, the radius of the lens.

It has already been observed, that only the pencil aAa' which emerges between b and b' , affords rays which appear to come from the caustic $IA'T$; so that an eye would see the luminous point A at this part of the caustic behind the glass only, within the space $bgKK'g'b'$. But if the eye were in the space CHK or $C'H'K'$, it would then receive rays which converge before the lens on the branches CH and $C'H'$, and which would in that case present an objective image hanging before the lens. That this image, however, would be much distorted, is apparent if the eye be supposed at that part of the axis where the extreme boundary rays CK and $C'K'$ intersect it; for there all the boundary rays meet, and the eye must see the luminous point as a circle whose radius would be nearly that of the lens. Besides which, the eye would also receive rays appearing to come from A' , and there the point A would again appear as a point: and thus the image of the point A would be resolved into a circle and a point.

In general, in the spaces CKH , $C'K'H'$, the eye would constantly observe two images of the point A , as here it would always receive two rays in different directions; and in fact a lens exhibits these two images very readily.

It is of importance to determine the space aa' on the lens which separates the rays actually coming to mutual intersection after passing through the lens, from those which would only intersect if carried backwards, so as to form the part $IA'T$ of the caustic. These two parts of the lens are bounded by the asymptotes cd and $c'd'$ of the caustic. The position of the asymptotes can be found only by a somewhat careful calculation. The angle which they form with the axis of the lens

is found in this case to be $7^{\circ} 21' 15''$. The arc $Bb = B'b'$, in the extremities of which the lens cuts the asymptotes, is $21^{\circ} 10' 56''$; and the ray Aa , which after the second refraction forms the asymptote cd , makes with the axis EF an angle $= 24^{\circ} 35' 7''$.

Figure 2.

The second figure, in which the same things are represented by the same letters, exhibits the course of the rays which proceed from a point A , out of the axis of the lens. If we suppose the point A to be the vertex of a pencil of rays which fall on the lens, the figure gives only those which lie in a plane passing through the luminous point and the axis of the lens. The caustic, even in this plane, has an un-symmetrical form; and the caustic which belongs to the cone is manifestly not generated by the revolution of this caustic about any line in its plane. In general, it is not easy to form a clear idea of the formation of the other parts of such a caustic: for if from A a ray out of the plane of the paper falls on the lens, it remains after the first refraction, which is at the surface CBC' in a plane which passes through it and the centre M of the sphere. It now proceeds in a straight line within the lens till it meets the second surface $CB'C'$. In order therefore to determine its course on emerging into the air, we must draw a plane through the centre M' of this spherical surface, and the direction of the ray within the lens, and the ray continues its course in this plane. As this ray has thus got into another plane, it is no longer evident what rays from A come to mutual intersection after being refracted the second time; since, without calculation, it is in general uncertain whether universally and at every point of the lens there are such rays as intersect after emerging.

The task, however, becomes simple as soon as the caustic is constructed which belongs to a pencil which has undergone only one refraction. Such a construction will hereafter be given in Part 3. Such a caustic, for instance, is obviously symmetrical, and two successive rays in a plane passing through the luminous point and the centre of the sphere, constantly intersect after

refraction. The cusp of the first caustic will again give rise to a cusp in the second caustic, which may be supposed to arise thus, viz. that the rays fall no longer from the luminous point on the second spherical surface, but from the caustic. And thus in our figure the point A' , which answers to A , is found on the line $m'N$, which is no other than the ray Amm' , which goes towards the centre M unrefracted, and which emerges at m' after refraction at the second surface. The calculation of the position of the point A' from that of A , is attended with difficulty. And as the first figure has been very carefully drawn, and this subject will come again under consideration in the third Part, I do not think it necessary to say anything more now in explanation of figure 2, as an attentive comparison of the two figures will give the necessary results.

PLATE X.

Before the lens DE is an object acb , which is seen by an eye A' behind the lens. The figure shews the effect of three luminous points upon the eye. The caustic answering to the point c , is $DCFE$ and $Hc'G$. In like manner, the caustic $DC'F'E$ and $H'a'G'$ belong to the point a , and the lines $DC''F''E$ and $b'''b'G''$ to the point b .

From a , the eye A' receives a pencil of rays whose axis is ah , which is refracted from h towards i , and here leaves the lens in such a form as it would have if the rays composing it proceeded from a' : therefore the eye sees the point a in a' . In precisely the same way it sees, by means of the refracted pencil $bkLA'$, the point b in b' , and the point c in c' . It sees, therefore, the rectilinear object acb in the position $a'c'b'$, which deviates from the rectilinear in the same way as both the other images $a''b''$ and $a'''b'''$, though less observably.

The visual angle $aA'b$ has altered into the greater angle $a'A'b'$; and it is obvious that the image $a'c'b'$ must appear magnified as compared with the object acb seen with the naked eye. But even if the visual angle were not increased, the

lens might still magnify; for the visual angle, and therefore the object, would be enlarged if the eye came nearer to it. It is known, however, that the distance of distinct vision is between 8 and 10 inches. At a distance less than this the object becomes indistinct, because the eye is no longer able to bring the rays which enter it to convergence; so that their points of convergence should fall on the retina. But if a lens DE is placed between the eye A' and the object ab , pencils from its different points will appear to come from the points $a'c'b'$, which lie at the distance of distinct vision: and thus the eye obtains a clear image of the object.

By the help of the preceding plate we may easily be convinced that it is only into the space DIE that rays from all points of the arrow can find their way; and that it is only here we can have a view of the entire object. But this space is still more limited for distinct vision, on account of the presence of such margin rays, CK , $C'K'$, as have been represented in figure 2, Plate IX, by which double images would be produced.

After what has been said it is not necessary further to explain how the eye A'' observes the arrow in the distorted form $a''c''b''$; and the eye A''' in the form $a'''c'''b'''$. These distortions seldom appear so considerable to the observer, as he has no distinct notion that the point c'' , for instance, lies really much nearer to him than a'' .

It is impossible to explain every circumstance represented on these figures: many new questions will be suggested to an attentive observer—but their solution will generally be obtained by inspection, or at all events by some simple figure of his own.

PLATE XI.

Figure 1.

The first figure of this plate also represents the course of the rays through a double convex lens DE ; but the luminous

point A is now at such a distance from it, that the caustic has become a curve DRE with a single branch. The luminous point A is assumed to be in the axis AH of the lens, and its two surfaces are of different curvature. Moreover, each ray is separated into two parts, the extreme red and violet. This separation has taken place in the lens, but is too minute there to be represented in the figure. Outside the lens, however, the red rays form an outer caustic DRE , which is separated from the inner violet caustic DVE . In order better to distinguish the rays from each other, the violet, marked v , have been extended a little beyond the red r . The extreme red rays Dr , Er , cut the axis of the lens in a : the distance of this point from the cusp R of the caustic, or the focus of the point A , is called the *longitudinal aberration*. This point of the caustic is also called the *central distance* of the rays, in order to distinguish it from the proper* focus; by which is generally understood, the cusp of that caustic which belongs to a luminous point at infinity.

If now the whole figure be made to revolve about the axis AH , it will describe the space occupied by the rays which fall on the lens from A . The points F and G , in which the caustic is met by the extreme marginal rays, describe by this revolution a circle, called the *least circle of aberration*, because it is the smallest circle through which all the refracted rays pass. There are no spherical lenses in which the longitudinal aberration, and therefore the circle of least aberration, is zero, i. e. which cause all rays proceeding from a point A to converge to a single point. But the radii of the spherical surfaces may be so determined, that these dimensions are a *minimum*; and this has been done with the lens here represented.

The thickness of the lens is assumed = 1, $AB = 37$; the refractive index $n = \frac{3}{2}$, and the focus R of the rays is at a distance = 4 from the point C . With these assumptions then, if the longitudinal aberration is to be a *minimum*, the radius r of the circle DBE must = 2.584, and the radius r' of DCE must

* The principal focus. The terms here employed are not quite the same as those commonly used in Cambridge.

= 7.971. A lens which has the property of making the longitudinal aberration a minimum, is called a *lens of the best form*. If the rays are parallel, and the focus lies so far behind the lens that the thickness of the lens compared with its distance may be neglected; in that case, the surface directed towards the rays must be about six times more curved than the other surface. A lens of this kind, in which the circle of least aberration is a minimum, is well adapted for a burning-glass. If the surface of less curvature were turned towards the sun, the same lens would concentrate the rays much more feebly, and thus be much less powerful. A figure in the third Part will exhibit this property of the lenses.

If through a small circular aperture rays be allowed to fall on a double convex lens, and the pencil of rays which has passed through the lens be received on a sheet of white paper behind it, a bright circle with a red rim is seen near the lens, which becomes smaller and brighter as the paper is withdrawn from the lens, and at length at the position FG appears without colour. If the paper is placed between F and R , a still smaller bright circle appears, which however is surrounded by a larger circle, more faint, and edged with blue. In the point R itself the bright circle is the smallest, and its circumference best defined. Beyond that point appears only a large, faint, and ill-defined circle.

These phænomena are fully explained by the inspection of the figure. It is seen also that the circles referred to must be brighter at the circumference than in the middle, as there the rays are more thickly crowded together. If in the equation

$$\frac{1}{\frac{m}{r} - \frac{1}{a}} + \frac{1}{\frac{m}{r'} - \frac{1}{a'}} = \frac{l}{n},$$

employed for Plate IX, the distance a of the luminous point be taken $= \infty$, the distance a' of the image from the other surface is given by the equation

$$a' = \frac{r' \left(r - \frac{m}{n} l \right)}{m \left(r + r' - \frac{m}{n} l \right)};$$

and if a' is put $= \infty$, we have

$$a = \frac{r \left(r' - \frac{m}{n} l \right)}{m \left(r + r' - \frac{m}{n} l \right)}.$$

From these equations it appears, that of the two foci of the lens, that is nearest to it which is opposite to the surface of least curvature. From the values above assumed we have $a' = 3.510$ and $a = 3.861$.

If then the point A be supposed to be to the left of the lens at infinity, its focus R falls at a distance 3.510 from the point C . If A approach nearer, R removes from the lens, but very slowly. For instance, its distance increases only from 3.510 to 4, while A moves from infinity to within 37 times the thickness of the lens. Now, however, the point R moves rapidly from the lens as the point A approaches nearer; for if the luminous point is only about 3.861 from the lens, the focus R is already at infinity. By a still nearer approach of the point A , the caustic assumes a different character, as is plain from Plate IX. Its cusp now falls on the left side of the lens, at first at infinity; it then comes nearer the point B as A approaches it, but still remains at a distance from it, in our case of about 0.696, even when A coincides with it.

If now the luminous point A move to the right of the lens in C , its focus also falls to the right at a distance $= 0.765$. So that an object which is placed close to the right of the lens, appears somewhat more magnified than if it were placed at B and observed from C . If, again, the luminous point be removed to a distance $= 3.510$ from the lens, its focus moves to infinity. With a greater distance of the point A , the caustic changes into the form here given; its cusp falls to the left of the lens, and approaches it within 3.861 if the luminous point be at infinity.

Figure 2.

This figure represents the effect of a pencil of rays, whose vertex A no longer lies in the axis HB of the lens. If this vertex is below the axis, the corresponding focus R is formed

above it. Since the same points are here marked by the same letters as the corresponding ones in figure 1, this figure requires no further explanation. The complete system of rays which corresponds to this representation, does not however arise from the revolution of the figure about any straight line, such as AR , which joins the points A and R , but does not much differ from such a solid of revolution, if the luminous point be not far from the axis.

Figure 3.

The two first figures are here made use of in order to illustrate the formation of an objective image behind a lens. The object aAa' is at the same distance from the lens as the point A in figures 1 and 2; only that it has been necessary to shorten the rays to obtain sufficient space behind the lens. The figure represents the caustics which correspond to the point, the middle, and the end of the arrow, and both the extreme red and the inner violet rays. The points of these caustics are respectively r, R, r , and v, V, v . The lines themselves are extended to the ends D and E of the lens, though in fact they terminate sooner. But to determine the point at which they stop requires a special calculation, which has been omitted because it is not essential.

The eye O in the axis of the lens receives pencils of rays whose vertices almost entirely coincide with the cusps of the caustics. The figure shews only the axes of these pencils. The observer thus sees a diminished, distorted, and red image of the arrow in the position rRr , and behind it a similar violet one, vVv . Between them lie a multitude of other images, with the various other prismatic colours. These images are nearly all concealed from the eye O , which therefore sees the object almost without any prismatic colouring.

The vertices of the red pencils of rays which the eye P receives, lie on the line pp' ; and in this position the arrow aa' also appears to the eye. The figure distinctly shews how the violet image is formed behind the red, and somewhat lower. Therefore the object will be seen from P with its natural colour only in the middle; it will appear red at the upper margin, and violet at the lower; for there the prismatic colours do not dis-

appear by total combination as they do in the centre. In exactly the same way the eye Q observes an image of the object in the position $q'q$; but this image is manifestly violet above at q' , and red at the vertex q .*

That moreover at the sides, even in small objects, prismatic colouring will also appear, is soon perceived, if it be considered that the caustic surfaces, which arise from points without the axis, are not symmetrical; and thus the vertices of the pencils of rays which reach the eye will not be in the same plane.

It is manifest that, on a white screen, in the neighbourhood of R , an inverted diminished image of the object may be obtained exactly with the convex lens. In other positions this can not be the case any more than in the concave lens; for here also the caustic surfaces of the luminous points would be intersected by the screen in circles, and not in points, as it must be to bring out a distinct image.

PLATE XII.

Figure 1.

This Plate shews the effect of a double-concave lens. The vertex of the incident pencil is at A , the axis of the lens $ABCD$. On this line therefore lie the centres of the spherical surfaces which bound the lens. Two consecutive rays, Aa , Aa' , are refracted, first towards ab , $a'b'$, and then after a second refraction emerge from the lens in directions bc , $b'c'$. If these lines are carried backwards they unite in d ; and thus an eye which received the minute pencil lying between the lines bc and $b'c'$, would there see the point A . And it is now sufficiently evident how, by the constant succession of such points as d , the caustic ISK is produced, and how we obtain an idea of the caustic surface and the entire system of rays which occupies the space bounded by that surface, by supposing the figure to revolve about the axis AD .

* This eye has been omitted.

The pencil AEF extends, after refraction behind the lens, into another $LGHM$, whose vertex is at O . If then the light which has passed through be received behind the lens upon a white screen, faint circles only are observed, which become larger and darker as the screen is removed from the lens. If such a concave glass, fixed in a dark ring,* be exposed to the sun's rays at a distance even of a few feet, the ring casts a shadow, such as it would if covered by an opaque disc; for the sun's rays which pass through the glass are even there extended over so wide a space, that their effect almost entirely disappears.

Figure 2.

The change which the caustic undergoes if the luminous point A is moved out of the axis, will at once appear from this figure. The cusp S of this caustic lies on the ray QR , which depends on the ray APQ ; and this ray has passed through the centre N of the circle EF , so as to enter the lens unrefracted, since it falls perpendicularly on its surface.

Figure 3.

Through a double-concave lens, or through any concave glass, the object is seen before it, diminished, in an upright position, and nearer to the observer. This figure, in connection with the two preceding, distinctly represents this effect of such lenses.

Of the object A_1, A, A_2 , the three caustics have been constructed which correspond to those three points. The eye O in the axis of the lens receives pencils of rays, whose vertices appear in a_1, a, a_2 ; and thus an image of the arrow is seen by it here, which is but slightly distorted. On the other hand, in other positions, as in P and Q , images b_1bb_2 and c_1cc_2 are seen of the straight arrow A_1AA_2 , observably distorted. In regard to the eye P the process is as follows: A_1E represents

* If the lens be not held in a ring, so as to have its edges completely covered, there will be an illumination of the screen due to the rays which emerge from the edges of the lens.

the axis of a small pencil whose vertex is in A_1 ; it enters the lens refracted into the direction EF , and leaves it at F again refracted, so that it reaches the eye P as a pencil whose vertex is b_1 . In like manner, $AGHP$ and A_1IKP represent the axes of pencils corresponding to points A and A_1 , and b , b_1 give the positions of the vertices of the pencils whose axes are represented by HP and KP . The eye P , therefore, observes an image of the arrow in the position b_1bb_1 . For the eye Q the figure gives the axes of the rays which give rise to the image c_1cc_1 .*

PLATE XIII.

Figure 1.

After having shewn in the preceding plates the effect of a convex and of a concave lens, our object in this is to exhibit the effect of a combination of the two. Such a combination is no other than the *Telescope of Galileo*, or the common opera-glass in its most simple form. As it was not necessary here to give the *best form* of such a telescope, but merely to explain in general the effect of such a combination of lenses, the forms of both glasses have been taken at pleasure.

The line AF represents the common axis of both lenses. From the point A of this axis a pencil falls on the convex lens HI , or, as it called, the object-glass of the telescope. Of this pencil a small part has been left out for want of space. This pencil would form a caustic surface behind the lens, whose cusp would lie in F , if the concave lens DE did not receive the rays refracted by the first lens, which now enter it between K and L , and leave it between M and N , in the form of a diverging pencil $MNPO$, which appears to proceed from the caustic MGN . On this caustic an eye which received the diverging pencil would see the point A .

* This eye is omitted.

Figure 2.

In this figure the effect of the two lenses is represented on a pencil whose vertex A is no longer on the common axis. It is here only necessary to remark, that the rays which fall between N and Q here undergo total reflexion and return into the lens, as shewn in the figure.

Figure 3.

This figure shews the formation of the *image* in such a telescope. Of the three points A_1, A, A_2 , of the object, three caustics have been found, whose cusps are in a_1, a, a_2 . The eye P receives pencils whose vertices are in these points. If the arrow is to be distinctly seen, the image a_1, a, a_2 , must stand at the distance of distinct vision of the eye; and as this is different for different eyes, the distances of the lenses must be altered, which is effected by means of two sliding tubes. The visual angle a_1Pa_2 is greater than the angle under which the object A_1AA_2 actually appears to the eye; therefore the combination of these two lenses not only makes the object more distinct, but also somewhat magnifies it. The degree in which it is magnified may be judged of thus, that the line FG^* gives the magnitude which the object would have if it appeared to the eye under the same angle as the image a_1aa_2 .

The eye Q receives from A_1 a pencil whose axis is represented by AH . This enters the lens in direction HI , leaves it in direction IK , and thus twice refracted at K and L comes into the eye in direction LQ . At L its rays leave the lens as though they all come from b_1 ; therefore the eye Q sees here an image of A_1 . To avoid crowding the figure with letters, the other pencils, which are easily understood, are not given. It will be seen at a glance how the image b_1bb_2 is formed for the eye Q . A combination of two convex lenses, employed in what is called the *Astronomical Telescope*, has been reserved for the *Third Part*.

* The points F and G are determined by producing the lines Ea_1, Ea_2 .

PLATE XIV.

It is, in fact, to the subject of this plate that this work is due. As teacher and examiner I had found that very few of those who study physical science possessed a distinct notion of the formation of the *rainbow*. This defect evidently arises among us, partly from the limited extent of mathematical knowledge, but partly also from the superficial method in which this subject is handled in the common treatises on physics.

I became convinced, however, that a careful graphical representation of this phenomenon would be sufficient to convey a satisfactory notion of the nature of it, even to those whose mathematical knowledge was small. Such a construction of the first rainbow answered my expectation so fully, that the thought occurred of applying the same method for the illustration of other optical phenomena.

The first figure represents by the circle, whose centre is M , the section of a sphere of water, or of a magnified drop of rain, by the plane of the paper. From an infinitely distant point of the diameter IG of this circle parallel rays fall upon it. Of these rays only the upper half $FGfg$ have been drawn, to allow space below for the emerging rays.

As soon as these rays fall on the drop they are separated into two parts, of which one is reflected, and so makes the sphere visible, (*i.e.* if the *scattered* rays are reckoned among these); the other part, refracted and resolved into colours, enters the sphere. In the figure only the *red* refracted rays have been retained. The mutual intersection of successive rays gives rise to a caustic which is intersected by the drop at c , and extends towards g , without reaching this point. All the rays which fall on the quadrant Gg , are compressed into the narrower space Ic , are here separated into two parts, of which one leaves the sphere refracted, the other returns into it by reflexion, and here extends over the arc GL . By this reflexion the rays again form a caustic cH . As soon as the rays have fallen on the arc GL , they are again separated: the reflected part of them is not included in the figure, but the refracted part is represented as being the most important for our object.

The part which emerges between l and d , forms here also, by mutual intersection of successive rays, the caustic lm , which at a greater distance from the drop nearly coincides with its asymptote de . This asymptote is formed by the ray cd , reflected at c , after it has left the drop refracted in d . In the figure the entire course of this ray is made conspicuous, as being of special importance, by the darker line $abcde$. The ray de also forms in fact the asymptote of the caustic, or meets its neighbouring ray at infinity, as appears from the consideration that the parallel rays nearest ab intersect after refraction at the surface of the sphere in c ,* and here on account of the symmetry of the sphere are reflected in the same way as they were incident, and therefore at d must leave the sphere again in a state of parallelism. If an eye receive the rays which emerge between G and d , it will have only a faint sensation of light from the luminous point, for these are less concentrated than the original rays, as is obvious from the figure, and are besides much weakened by their resolution into colours, and frequent reflexion and refraction. On the other hand, the rays which emerge between d and l are much compressed, and at the edge of the caustic lm , which at a greater distance from the drop is almost confounded with its asymptote, are sufficiently powerful to affect the eye strongly.

The effect on the eye will be greater on this account also that in the space $kilm$ they are doubled over each other, for the rays which fall on the drop between ab and fg are refracted towards the arc ch , and here so reflected as again to form a caustic in the sphere extending from l to h , but which is not sufficiently distinct in the figure, as in it the rays are made to emerge too far apart.

We have hitherto only followed the course of rays in a plane passing through the luminous point and the centre of the sphere. But if we suppose the construction carried out in this plane to

* Two rays refracted near to b intersect in the point c because it is a point on the caustic cg . Hence the inclination of the two rays after reflexion at c is the same as it was before incidence at c . Also, since $cd = cb$, the effect of refraction at d is exactly the same as that of the refraction at b ; consequently, on emerging at d two contiguous rays will be parallel.

revolve about the axis IF , the semicircle GgI passes over the upper surface of the drop, and the line fg passes over the cylinder occupied by parallel rays which falls on the sphere. The caustic lines then describe caustic surfaces, at which the reflected and refracted rays of the incident cylinder intersect. The line de by this revolution passes over the surface of that cone which always approaches nearer to the caustic described by the curve line, the further both are from the sphere, but never coincides with that caustic.

This cone is thus the asymptotic cone of the caustic surface. Accordingly, an eye placed at the requisite distance from the surface of this cone receives from the luminous point red light. For the red rays, the side of the asymptotic cone makes with the axis of it an angle of $42^\circ.2'$, and this is the value of the angle which de makes with IF or with ne .

If an eye be supposed placed in a fixed position in the horizontal line ne which runs parallel with IF , and the drop in such a position that IF coincides with ne , the eye will then receive but a faint impression derived from rays which emerge immediately below G ; but if the drop move perpendicularly from the line ne , so that IF is always parallel to this line, rays will fall upon the eye, emerging with continually increasing density, but still not sufficiently powerful to produce a sensation of light.* It is only when the globule in the way stated is raised so far that the eye is met by the asymptotes of the caustic, or is on the asymptotic cone, that a red image of the luminous point (which is infinitely distant in the direction FG) is seen. The word *image* here is used in a very general sense, as signifying only a sensible perception of the object by the eye.

But if the drop be raised still higher above the horizontal line ne , the eye will no longer be struck at all by the rays of the luminous point.

Suppose now a plane perpendicular to the line ne to be thickly covered by drops of rain, and a luminous point from

* At the distance of the eye the divergent rays which emerge from the drop are so widely separated that they cease to produce a *sensible* impression of light. See p. 33.

infinity in the direction ne to throw red rays upon this plane, then the eye A will see on this plane a red circle whose centre is in the line ne and whose radius will have an apparent magnitude of $42^\circ.2'$; for all the spheres which are on a line making this angle with the line ne make an impression on the eye. The drops within this circle will not indeed make a vivid impression, but always more so than those which lie without the circle, for from these the eye receives no rays coming from their interior, but only those which are reflected from their surface, so that the inner part of this circle must appear brighter than the outer. If instead of a single plane the entire space before the eye were supposed filled with drops of rain, then all the globules lying on the surface of the cone generated by the revolution of the angle den about the fixed line ne produce a sensation of light; the eye would manifestly see only a circle, as it is situated at the vertex of the cone, but this circle would increase with the distance of the rain-drops.

We have assumed for the red rays the index $\frac{4}{3}$, that for the violet rays is greater by about $\frac{1}{81}$. On this account the angle which the side de of the cone makes with the axis IG is diminished to $40^\circ.16'$, and the angle den would be also $40^\circ.16'$. Hence the eye at A perceives on a plane covered with drops illuminated with violet rays from a point at infinity, a violet circle whose apparent radius is $40^\circ.16'$.

But if from such a point white light falls upon a rain-drop, it will be resolved into prismatic colours; and if the drop is $40^\circ.16'$ above the line ne , then the violet rays fall upon the eye A ; if the drop be higher, the other prismatic colours reach the eye, till at length, if the drop be $42^\circ.2'$, only the red rays come into the eye. If now a point at infinity illuminates with white light a space occupied by drops of rain, and a straight line An be drawn through the luminous point and the eye A , then all the drops which lie on the surface of a cone whose side ed forms with the axis en , an angle $= 40^\circ.16'$, send violet rays to the eye, and from those which are on the surface of a cone whose side makes $42^\circ.2'$ with the axis, it receives red rays: between these

two conical surfaces lie the innumerable other rays by which the other prismatic colours reach the eye.

These conical surfaces are intersected in circles by planes perpendicular to the axis; therefore a single luminous point at a great distance would produce on a screen of rain a prismatically coloured circle, whose apparent angular breadth would be $42^{\circ}.2' - 40^{\circ}.16' = 1^{\circ}.46'$: the appearance is not interfered with by the circumstance that the drops are in falling motion, as the place of a falling drop is immediately supplied by another.

But such a circle is produced by every point of the sun's disc. If one set of rays, the red for example, alone proceeded from the sun's surface, a red circle would be seen on the screen of rain, whose apparent breadth would be $32'$, *i.e.* the apparent magnitude of the sun's diameter.

But as the sun emits white light, every prismatic colour gives rise to a circle of the same breadth; these circles partly overlap and increase the above-stated breadth of the prismatic rings by about $32'$, so that the whole apparent rim of the rainbow is $1^{\circ}.46' + 32' = 2^{\circ}.18'$, or rather greater than 4 times the sun's apparent diameter.

These numbers would have to be somewhat modified if the form of the rain-drops were not reckoned exactly spherical, and indeed careful measurements make the apparent radius of the rainbow about half a degree less than the angle which the asymptote of the caustic makes with the incident ray. Mr. Airy has explained this phenomenon according to the theory of undulations thus, that the greatest intensity of light at the caustics does not lie in the asymptotes.*

When the sun is in the horizon, the rainbow appears only as a semicircle, for the axis of the cone whose vertex is in the eye falls on the cloud in the horizon. If, on the other hand, the sun has more than $42\frac{1}{2}$ degrees' elevation above the horizon, the first or principal rainbow can in general be no longer formed.

* See *Cambridge Philosophical Transactions*, vol. vi. p. 391. The words used by Mr. Airy are, "The maximum illumination does not take place at the Geometrical Caustic."

Figure 2.

It is well known that over the first or principal rainbow a space appears somewhat darker than the rest, and then follows a second broader but much fainter rainbow, whose colours are in reversed order, so that it begins with the red and is bounded without by the violet rays. The second figure explains fully the rise of this rainbow. Between the lines rs and gh parallel rays from infinity fall on the spheres which enter them refracted. Of these refracted rays the red only are retained as before. These rays are concentrated on the arc oIp , are here separated into two parts, of which the one leaves the drop and forms the caustic oKp , the other returns into the drop, and at the other surface is again separated, whereby the emerging refracted part, here left out, gives rise to the principal rainbow, but the reflected part within the drop forms a caustic with cusp L , and then emerges refracted. This emerging part now gives rise to the second rainbow, for it also produces a caustic, which appears faintly marked in the figure (between l and f) on account of the great separation of the rays.

The individual ray which forms the asymptote of this caustic, observed by the eye at A , is marked with the letters $abcdef$. Whereas in the formation of the first rainbow, that part only of the sun's rays is effective which falls upon the upper half of the drop, the rays which enter the under part give rise to the second rainbow; the angle efn , which the asymptote ef makes with the line fn parallel to the axis IG , is for the red rays $50^{\circ}.59'$, for the violet $54^{\circ}.10'$.

After the explanation of the first rainbow, it is only needful to remark that, in accordance with the figure and calculation, at an elevation of $50^{\circ}.59'$ above the line which passes through the sun's centre and the eye of a spectator with his back to the sun, the centre part of a red circle $32'$ broad will appear on a screen of rain, while at $54^{\circ}.10'$ will be seen the middle part of a violet circle of the same breadth. Between both lie other exactly similar circles, illuminated by the other prismatic colours. The second rainbow has thus a breadth of

$$54^{\circ}.10' - 50^{\circ}.59' + 32' = 3^{\circ}43',$$

and is at its lower red part distant from the extreme red of the principal rainbow, $70^{\circ}.25'$. The drops which are between both rainbows send to the eye no rays whatever of those which come from within them, but only those reflected from their surface; therefore this part appears darker than the rest.

If the sun's rays were of such intensity as not already to have been too much weakened by the many reflexions and refractions, it is manifest that more rainbows might be formed, whose brightness however would be continually diminished.

The third rainbow, for example, would surround the sun at a distance of 37° , and at 42° end with the violet part, and thus have a breadth of 5° . It would, on account of its breadth and faintness, be scarcely visible, while the nearness of the sun would make it still more indistinct even if the conditions for its formation existed.

APPENDIX.

NOTES ON PLATE I.

(1) THE equation of the caustic in figure 3 is easily found. Thus if H be the origin of coördinates, HA the positive direction of the axis of x , HE that of y ; if also HA be put equal to a , HA' to α , and β be the distance from H at which the incident ray meets the surface of the medium, the equation of the reflected ray will be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 \dots \dots \dots (1),$$

and we shall have, by the law of refraction,

$$a^2 + \beta^2 = \mu^2 (\alpha^2 + \beta^2),$$

or

$$\mu^2 \alpha^2 + (\mu^2 - 1) \beta^2 = a^2 \dots \dots \dots (2),$$

μ being the index of refraction for the medium below the line DE .

Now if x and y be the coordinates of a point on the caustic, they will remain constant when α and β vary by an indefinitely small quantity. Hence, by differentiating (1) and (2), we shall have the following relation between x , y , α , and β :

$$\frac{x}{\alpha^2} d\alpha + \frac{y}{\beta^2} d\beta = 0,$$

$$\mu^2 \alpha d\alpha + (\mu^2 - 1) \beta d\beta = 0;$$

therefore

$$\begin{aligned} \frac{\frac{x}{\alpha^2}}{\mu^2 \alpha} &= \frac{\frac{y}{\beta^2}}{(\mu^2 - 1) \beta} \dots \dots \dots (3), \\ &= \frac{1}{\alpha^3} \text{ by (1) and (2).} \end{aligned}$$

Therefore

$$\frac{x^3}{a^3} = \frac{\mu^3 x^3}{a^3},$$

$$\frac{y^3}{b^3} = \frac{(\mu^3 - 1) y^3}{a^3};$$

therefore, by equation (1),

$$\mu^{\frac{3}{2}} x^{\frac{3}{2}} + (\mu^3 - 1)^{\frac{1}{2}} y^{\frac{3}{2}} = a^{\frac{3}{2}},$$

or

$$\left\{ \frac{x}{a} \right\}^{\frac{3}{2}} + \left\{ \frac{y}{\sqrt{(\mu^3 - 1)}} \right\}^{\frac{3}{2}} = 1 \dots \dots \dots (4).$$

Now if a_1 and b_1 be the semiaxes of an ellipse, the equation of its evolute is

$$\left\{ \frac{x}{\frac{a_1^2}{a_1} - b_1^2} \right\}^{\frac{3}{2}} + \left\{ \frac{y}{\frac{a_1^2}{a_1} - b_1^2} \right\}^{\frac{3}{2}} = 1 \dots \dots \dots (5).$$

Comparing this with (4), it appears that the caustic curve is the evolute of an ellipse whose eccentricity is $\frac{1}{\mu}$, and whose semi-major-axis is μa .

In the case of water, $\mu = \frac{4}{3}$, and therefore the equation of the curve represented in figures 3 and 4, is

$$\left\{ \frac{x}{\frac{3a}{\sqrt{7}}} \right\}^{\frac{3}{2}} + \left\{ \frac{y}{\frac{4}{3}a} \right\}^{\frac{3}{2}} = 1.$$

If in figure 5 the distance of the point A from the surface of the medium be put equal to a , and the same axes of coordinates be adopted, a similar investigation will shew that the equation of the caustic curve is in this case

$$\left(\frac{x}{\mu a} \right)^{\frac{3}{2}} - \left\{ \frac{y}{\frac{\mu a}{\sqrt{(\mu^3 - 1)}}} \right\}^{\frac{3}{2}} = 1,$$

which is the equation of the evolute of a hyperbola whose eccentricity is μ , and whose semi-major-axis is $\frac{a}{\mu}$.

In the case of water the equation becomes

$$\left\{ \frac{x}{4a} \right\}^{\frac{4}{3}} - \left\{ \frac{y}{4a} \right\}^{\frac{4}{3}} = 1.$$

It thus appears that for the same medium the caustics corresponding to different points are all similar curves, so long as the points are all within or all on the outside of the medium. The dimensions of the curves meanwhile vary for different distances of the luminous point from the surface of the medium.

(2) Figure 4 explains the way in which an object under water becomes visible to an eye above the water; and also very clearly shews the change of form which the object appears to undergo in consequence of refraction. Figure 6, by constructions of a similar kind, shews the form which an external object assumes to an eye under water. A consideration of figures 3, 4, 5, and 6, will enable us to understand the *general* appearances presented to an eye under water, or within any refracting medium.

It will be taken for granted that if a ray of light can be proved to take a particular course in passing from any one point of space to any other, a ray of light proceeding from this latter point to the former one will follow exactly the same course reversed.

Now in the explanation of figure 3, it has been proved that rays of light issuing in every direction from the point *A*, and after undergoing certain modifications (which were explained), eventually reached every point of space. For it was shewn that all the rays between *AD* and *AE* emerge from the water after undergoing refraction at its surface, and so pass to all external points of space. All the rays also beyond *AD* and *AE* after incidence upon the surface were shewn to be totally reflected and turned back into the medium; whilst all points within the medium are of course reached by rays which proceed in straight lines from *A*.

Suppose then that an eye is placed at the point *A* in figure 3. The lines which have been considered as rays proceeding from *A* may now be regarded as the axes of pencils which reach the

eye from all points of space. Since, therefore, of the rays which issued from *A*, those only which occupied the space *DAE* could emerge, and having emerged were shewn to proceed to all external points of space; conversely, the axes of pencils proceeding from all external points must, in order to reach the eye, enter the medium at points situated between *D* and *E*. For instance, the ray *AK* proceeds after refraction in the direction *KL*; therefore, by the principle we have assumed, the axis of a pencil proceeding from *L* must be refracted at *K* in order to reach the eye at *A*. An object at *L* will thus be seen in the direction *AK*.

All external objects then to an eye under water will appear to be gathered into a cone, whose semi-vertical angle is $\sin^{-1}(\frac{2}{3})$. Those objects which are nearly opposite to the eye in the direction *AF* will appear to be very little displaced from their real position, whilst objects near the horizon will be crowded together in the neighbourhood of the cone's bounding surface.

In like manner it will be seen that all objects under water and beyond the lines *DF*, *EG*, will be twice seen; once by direct rays, and again by rays totally reflected at the surface.

It is worthy of remark that all objects situated on the lines *DF*, *EG*, will be seen by reflexion very nearly in the same direction as external objects situated on the horizon. In this way a fish may appear to be swimming amongst the grass or in the branches of a shrub.

A little consideration will also shew that a stick or any other object partly immersed in the water beyond the points *D* and *E* will appear to be separated into two parts. The part which is above the water will appear, with other external objects, within the cone above mentioned. The part which is immersed in the water will be seen by direct vision in its true position, and also by total reflexion, as it would be if it were viewed in contact with a polished plane mirror.

Objects under water in the space between the eye and the lines *DF*, *EG*, will be seen only by direct vision; for although the rays which proceed from them are partially reflected at the surface, yet so large a portion of the light emerges that the image formed by reflected rays is too faint to be distinguished.

The rays in the figure are confined to one plane. All that has been said applies equally to any plane through AI , and the whole effect may be represented by supposing the figure to revolve about the line AI .

An expert diver (as I have been informed) may see these phenomena whilst he is bathing. Of course he will only obtain a confused impression of the true appearance, because the surface of the water will be in a state of greater or less agitation.

NOTES ON PLATE II.

(1) The following investigation of the equation of the Caustic by Reflexion at a circle, as represented in this figure, is the same in principle as the one given by Lagrange in the *Mem. de Turin*, and to which attention was called by Mr. P. Smith, of the British Museum, in the *Cambridge and Dublin Mathematical Journal*, vol. II. p. 236.

Let the origin and axes be taken as in the text, and let r be the radius of the circle, a the distance of the luminous point from the centre.*

Let α be the angle of incidence of any ray proceeding from C , θ the angle which the radius, drawn to the point of incidence, makes with the axis of x ; then $\theta - \alpha$ will be the angle which the reflected ray makes with that axis.

Now the length of the perpendicular from the origin upon the reflected ray is $r \sin \alpha$; the equation of this ray is therefore

$$y \cos(\theta - \alpha) - x \sin(\theta - \alpha) = r \sin \alpha \dots\dots\dots (1).$$

We have also the relation

$$r \sin \alpha = a \sin(\theta + \alpha) \dots\dots\dots (2).$$

Eliminating α from these equations, and reducing, we obtain for the equation of the reflected ray,

$$ax \sin 2\theta - ay \cos 2\theta - rx \sin \theta + ry \cos \theta = ar \sin \theta \dots (3).$$

* The reader is requested to draw a figure for himself.

Now, at the point where this ray touches the caustic x and y remain constant, whilst θ varies by an indefinitely small quantity. Let us for convenience write $f \cos 2\beta$ for x , and $f \sin 2\beta$ for y . The last equation becomes

$$af \sin 2(\theta - \beta) - rf \sin(\theta - 2\beta) = ar \sin \theta.$$

Again, let $\theta = \phi + \beta$, and therefore $\theta - 2\beta = \phi - \beta$;

$$\begin{aligned} \therefore af \sin 2\phi &= rf \sin(\phi - \beta) + ar \sin(\phi + \beta) \\ &= r(a+f) \cos \beta \sin \phi + r(a-f) \sin \beta \cos \phi : \end{aligned}$$

dividing both sides of the equation by $arf \sin \phi \cos \phi$,

$$\left(\frac{1}{f} + \frac{1}{a}\right) \frac{\cos \beta}{\cos \phi} + \left(\frac{1}{f} - \frac{1}{a}\right) \frac{\sin \beta}{\sin \phi} = \frac{2}{r}.$$

$$\text{Let} \quad \left(\frac{1}{f} + \frac{1}{a}\right) \cos \beta = \frac{2P}{r},$$

$$\text{and} \quad \left(\frac{1}{f} - \frac{1}{a}\right) \sin \beta = \frac{2Q}{r};$$

$$\text{then} \quad \frac{P}{\cos \phi} + \frac{Q}{\sin \phi} = 1 \dots\dots\dots (4).$$

Now, if we suppose ϕ to vary whilst P and Q remain constant, it is the same thing as supposing θ to vary whilst x and y remain constant. But upon this supposition we obtain, by differentiating equation (4),

$$\begin{aligned} \frac{P}{\cos^2 \phi} &= \frac{Q}{\sin^2 \phi} \\ &= (P^{\frac{2}{3}} + Q^{\frac{2}{3}})^{\frac{3}{2}}; \end{aligned}$$

and substituting in (4),

$$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1.$$

Raising this to the third power,

$$P^2 + Q^2 + 3P^{\frac{2}{3}}Q^{\frac{2}{3}}(P^{\frac{1}{3}} + Q^{\frac{1}{3}}) = 1;$$

$$\therefore 1 - (P^2 + Q^2) = 3P^{\frac{2}{3}}Q^{\frac{2}{3}};$$

$$\therefore \{1 - (P^2 + Q^2)\}^3 = 27P^2Q^2.$$

$$\begin{aligned}
\text{Now } P^2 + Q^2 &= \frac{r^2}{4} \left\{ \frac{1}{f^2} + \frac{1}{a^2} + \frac{2}{af} \cos 2\beta \right\} \\
&= \frac{r^2}{4a^2 f^2} (a^2 + f^2 + 2af \cos 2\beta) \\
&= \frac{r^2}{4a^2 (x^2 + y^2)} (a^2 + x^2 + y^2 + 2ax), \\
1 - (P^2 + Q^2) &= \frac{(4a^2 - r^2) (x^2 + y^2) - 2ar^2x - a^2r^2}{4a^2 (x^2 + y^2)}, \\
\text{and } P^2 Q^2 &= \frac{r^4}{16} \left(\frac{1}{f^2} - \frac{1}{a^2} \right)^2 \sin^2 \beta \cos^2 \beta \\
&= \frac{r^4}{16} \cdot \frac{(f^2 - a^2)^2 f^2 \sin^2 2\beta}{4a^2 f^6} \\
&= \frac{r^4 a^2 (x^2 + y^2 - a^2)^2 y^2}{\{4a^2 (x^2 + y^2)\}^3}.
\end{aligned}$$

Therefore we derive the following equation of the Caustic :

$$\{(4a^2 - r^2) (x^2 + y^2) - 2ar^2x - a^2r^2\}^3 = 27a^2r^4y^2(x^2 + y^2 - a^2)^2.$$

(2) The equation of the Asymptotes of the Caustic may be found by the consideration, that it touches the caustic at an infinite distance. Now, since the caustic is formed by the intersection of consecutive reflected rays, if two of such rays intersect at an infinite distance they are parallel. Now $\theta - \alpha$ is the inclination of any reflected ray to the axis; so that if two consecutive rays be parallel, $\theta - \alpha$ will not vary when θ and α vary by an indefinitely small quantity. We must therefore have $\frac{d\theta}{d\alpha} = 1$ in the case of the ray which, after reflexion, becomes an asymptote.

Differentiating (2) on this supposition, we obtain

$$r \cos \alpha = 2a \cos(\theta + \alpha);$$

$$\text{hence } \tan \alpha = \frac{2a \cos \theta - r}{2a \sin \theta},$$

$$\text{and from (2), } \tan \alpha = \frac{a \sin \theta}{r - a \cos \theta}.$$

Equating these values of $\tan \alpha$, and reducing,

$$2a^2 + r^2 = 3ar \cos \theta \dots\dots\dots (5),$$

an equation which determines the point of incidence of the ray which, after reflexion, becomes the asymptote.

In order to obtain the equation of the Asymptote, we must substitute in (3) the value of θ obtained from (5). Now (3) may be written

$$y (a \cos 2\theta - r \cos \theta) = \sin \theta \{x (2a \cos \theta - r) - ar\},$$

$$\text{or} \quad y = \frac{\sin \theta (2a \cos \theta - r)}{a \cos 2\theta - r \cos \theta} \left(x - \frac{ar}{2a \cos \theta - r} \right).$$

On reducing this equation by means of (5), the equations of the asymptotes will be found to be

$$y = \pm \frac{4a^2 - r^2}{8a^2 + r^2} \sqrt{\left(\frac{4a^2 - r^2}{r^2 - a^2} \right)} \left(x - \frac{3ar^2}{4a^2 - r^2} \right) \dots\dots (6).$$

These equations at once shew that the distance MQ is equal to $\frac{3ar^2}{4a^2 - r^2}$. (See p. 11.)

(3) The expression for $\sin m$, in p. 11, may be readily found. Putting $m = \theta - \alpha$, and substituting, we have

$$r \sin \alpha = a \sin (m + 2\alpha),$$

$$r \cos \alpha = 2a \cos (m + 2\alpha).$$

Multiplying the first equation by 2, squaring and adding,

$$r^2 (1 + 3 \sin^2 \alpha) = 4a^2,$$

$$\therefore \sin^2 \alpha = \frac{4a^2 - r^2}{3r^2}.$$

Multiplying the first equation by $2 \cos 2\alpha$, the second by $\sin 2\alpha$, and subtracting,

$$2r \cos 2\alpha \sin \alpha - r \sin 2\alpha \cos \alpha = 2a \sin m,$$

$$\text{or} \quad -r \sin^3 \alpha = a \sin m;$$

$$\therefore \sin m = -\frac{r}{a} \sin^3 \alpha$$

$$= \pm \frac{1}{ar^2} \left(\frac{4a^2 - r^2}{3} \right)^{\frac{3}{2}}.$$

The two signs shew that there are two asymptotes equally inclined on opposite sides to the axis of x .

When $a = \frac{r}{2}$, *i. e.* when the luminous point is half-way between the centre and the circumference of the reflecting circle, $\sin m = 0$, and the asymptotes coincide with the axis of x .

Equation (6) shews that there can only be an asymptote when a lies between $\frac{r}{2}$ and r . When $a = r$, the luminous point is on the circumference. In this case equation (6) assumes the form $x = r$, and is the equation of the ray perpendicular to the axis which undergoes no reflexion. The caustic is in this case the cardioid.

(4) If we describe a circle with centre M and radius MC , it is plain that the ray Ce , and its reflected ray ed , both touch this circle, and contain between them a greater angle than any other such pair of rays, since every other incident ray is nearer to the centre than Ce . Every other ray through C cuts off a certain arc from this circle, and the corresponding reflected ray cuts off an equal arc, because by the law of reflexion every incident ray is at the same distance from the centre as its reflected ray. If, therefore, we consider any ray incident near to the ray Ce , after reflexion, it will cut the circle in two points very near to the point d ; and two such rays to the left of Ce will, after reflexion, intersect each other in a point within the circle. Suppose these rays to approach to the position Ce , the above-named points continually approach each other and the point d . Hence in the limit the ray ed and its immediately consecutive ray will intersect in d , and therefore d will be a point on the caustic.

Since also the angle Ced is greater than the angle between any other pair of incident and reflected rays, a little consideration will shew that a pair of rays to the left of Ce will intersect within the circle whose radius is MC , and a pair of rays to the right of Ce will intersect without that circle. Hence the point d will be nearer to e than the points of the caustic on either side near to it. The point d will therefore be a cusp of the caustic; and since every reflected ray is a tangent, the ray ed will be a tangent at the cusp.

(5) To prove more strictly that two consecutive rays in the neighbourhood of e intersect in the point d , let us suppose that Ce' is a ray incident near to the point e , and that the corresponding reflected ray $e'd'$ intersects ed in the point d' . Draw the normals Me, Me' ; then it is easily proved, from the equality of the angles of incidence and reflexion, that

$$\angle eCe' + \angle ed'e' = 2 \angle eMe'.$$

This equation is true in the limit when the incident rays become consecutive. But in the limit the angles of incidence become equal; and if we put the angle $CeM = \phi$, and suppose x to be the limiting value of ed' , then the $\angle eCe' = \frac{ee' \cos \phi}{Ce}$, the $\angle ed'e' = \frac{ee' \cos \phi}{x}$, and the $\angle eMe'$ is always equal to $\frac{ee'}{r}$;

$$\therefore \frac{1}{x} + \frac{1}{Ce} = \frac{2}{r \cos \phi};$$

but $r \cos \phi = Ce$, and $\therefore x = Ce$.

But $de = Ce$, and therefore d is a point of the caustic.

(6) Again, if x' and ϕ' be values of x and ϕ for the ray immediately consecutive to Ce , we shall have in the limit

$$\frac{\cos \phi'}{x'} + \frac{\cos \phi}{Ce} = \frac{2}{r};$$

$$\text{but } \frac{\cos \phi'}{x'} = \frac{\cos \phi + \delta \cos \phi}{x + \delta x} = \frac{\cos \phi}{x} \left(1 + \frac{\delta (\cos \phi)}{\cos \phi} - \frac{\delta x}{x} \right) \text{ nearly.}$$

Therefore, substituting, we have

$$\frac{\delta (\cos \phi)}{\cos \phi} = \frac{\delta x}{x},$$

therefore

$$\delta x = -x \tan \phi \cdot \delta \phi;$$

but $x \tan \phi = a$, therefore

$$\delta x = -a \delta \phi.$$

Consequently, if ϕ diminishes x increases in the immediate neighbourhood of e . But, by what has been said, ϕ does diminish whichever way we start from e , and therefore the point d is a cusp of the caustic.

NOTES ON PLATE VIII.

1. It does not appear that the curve which results from the intersection of rays after refraction through a prism has an asymptote. For on the right of the line *rcsu* in figure 1, the curve terminates at the face of the prism with that ray which is first totally reflected. On the left of that line, if ϕ and ϕ' be the angles of incidence and refraction at the first surface, ψ and ψ' the angles of emergence and incidence at the second, if also μ be the index of refraction, and i the angle, we have

$$\phi' + \psi' = i \dots\dots\dots (1),$$

$$\sin \phi = \mu \sin \phi' \dots\dots\dots (2),$$

$$\sin \psi = \mu \sin \psi' \dots\dots\dots (3).$$

Now if there be an asymptote to the caustic it will arise from the circumstance that two consecutive rays are parallel on emerging from the prism: in other words, since ψ is measured from a fixed direction (the normal to the face *DE*), we shall determine the asymptote by the consideration that ψ does not vary when ϕ receives a small increment, *i.e.* $\frac{d\psi}{d\phi}$ must vanish.

Differentiating (2) and (3),

$$\frac{\cos \phi}{\cos \psi} \cdot \frac{d\phi}{d\psi} = \frac{\cos \phi'}{\cos \psi'} \cdot \frac{d\phi'}{d\psi'},$$

but

$$\frac{d\phi'}{d\psi'} = -1 \text{ by (1);}$$

therefore

$$\frac{d\psi}{d\phi} = - \frac{\cos \phi \cos \psi'}{\cos \phi' \cos \psi}.$$

Now for a ray which emerges from the prism ψ' can never exceed $\sin^{-1} \left(\frac{1}{\mu} \right)$, and therefore $\cos \psi'$ can never vanish. On the other hand, $\cos \phi$ can vanish only when $\phi = 90^\circ$, and this can only be the case when the point of incidence of the incident ray is removed to an infinite distance from the origin of light, *i.e.* when the side of the prism is infinitely extended. The curve therefore has no asymptote.

(2) The second figure of this plate shews very clearly that the axes of the small coloured pencils which proceed from any point of an object enter the eye in different directions. It will thus be seen that the image of a point upon the retina is a pure spectrum. For since the eye is achromatic for every kind of light, it is able to bring pencils of every colour to a point of convergence on the retina. Moreover it appears from figure 2, Plate XI, that if the axes of any number of pencils fall in different directions upon a lens, the points to which the rays in those pencils converge will be different. We conclude therefore that every coloured pencil comes to a different point on the retina, because its axis falls in a different direction upon the lens of the eye, so that light of every kind reaches the retina in an unmixed state, and the impression produced is a series of points all of a different colour.

If the point from which the light proceeds and the eye be in the same plane perpendicular to the edges of the prism, the image of the point will manifestly be a coloured line in that plane.

NOTES ON PLATE IX.

(1) To find the formula on page 23, we proceed as follows :

Draw the radii aM and bM' , and suppose x to be the point in which ba produced cuts the axis of the lens : then, by the law of refraction,

$$\begin{aligned} n &= \frac{\sin AaM}{\sin Mab}, \\ &= \frac{\sin AaM}{\sin AMa} \cdot \frac{\sin XM a}{\sin XaM}, \\ &= \frac{AM}{Aa} \cdot \frac{Xa}{XM}. \end{aligned}$$

If now we suppose the angle aAM to diminish, and a to approach to B , in the limit $Xa = A_1B$, $XM = A_1M$, and $Aa = AB$; therefore

$$n.AB.A_1M = AM.A_1B,$$

if A_1 be the limiting position of point X .

In like manner, if Y be the point in which cb produced meets the axis of the lens, the limiting position of Y is A' ; and we shall have, by similar reasoning,

$$nA'B'.A_1M' = A'M'.A_1B'.$$

Now if in each of these equations we suppose positive lines to be measured from the centre towards the circumference, we shall have

$$\begin{aligned}\frac{m}{r} - \frac{1}{a} &= -\frac{n}{A_1B'}, \\ \frac{m}{r'} - \frac{1}{a'} &= \frac{n}{A_1B'}; \\ \therefore \frac{1}{\frac{m}{r} - \frac{1}{a}} + \frac{1}{\frac{m}{r'} - \frac{1}{a'}} &= \frac{l}{n}.\end{aligned}$$

(2) The investigation of the position of the asymptotes of the caustic formed by refraction through a double-convex lens is very difficult. After having tried the general problem I confess that I am unable to give a solution. The equations are subjoined in case any one more patient or more skilful should feel disposed to attempt this very unpromising labour.

If ϕ and ϕ' be the angles of incidence and refraction at the first surface; ψ and ψ' the angles of emergence and incidence at the second; if θ and θ' be inclinations to the axis of the lens of the radii at the points of incidence and emergence, and b the distance MM' , we shall have

$$\sin \phi = \mu \sin \phi' \dots \dots \dots (1),$$

$$\sin \psi = \mu \sin \psi' \dots \dots \dots (2),$$

$$\theta + \theta' = \phi' + \psi' \dots \dots \dots (3),$$

$$r \sin \phi = (a + r) \sin (\theta + \phi) \dots \dots \dots (4),$$

$$r \sin \phi' - r' \sin \psi' = b \sin (\theta' - \psi') \dots \dots \dots (5),$$

and the condition will be that $\theta' - \psi$ shall not vary when the incident ray changes its direction through an indefinitely small angle.

NOTE ON PLATE XIII.

If the reader bears in mind what has been said on Plate XI, he will understand that the image seen by Galileo's Telescope will be indistinct on account of the separation of common light into its component colours. If we confine our attention to the red and violet pencils, we shall see that a red and violet caustic is formed by the lens *HI*; and that to each of these caustics corresponds another caustic after the rays have passed through the lens *KN*. Caustics of other colours will lie between these extreme ones.

In consequence of this separation of the light the eye will see a multitude of images, of which the whole effect will be a single white image slightly coloured at its borders.

The effect of colour in Galileo's Telescope is slight, because the refraction at the two lenses takes place in opposite directions. It thus happens that after a ray of light has been dispersed by the first lens, the violet part falls nearer to the axis of the second lens than the red part, and therefore it undergoes less refraction than the red ray, and on emergence the separation is less than it was before; the caustics therefore after refraction through the second lens are very close together, and the indistinctness produced is in a corresponding degree small.

This reasoning applies only to a direct pencil like that in figure 1. The same considerations however approximately hold with regard to a pencil of small obliquity.

The defect arising from the dispersion of light is far more serious in the case of the common Astronomical Telescope, and requires a more particular consideration.

